

Counting Photons Differently: A Combinatorial Re-Reading of Bose's 1924 Derivation of Planck's Law

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Abstract- In 1924 Satyendra Nath Bose, then at the University of Dacca, derived Planck's radiation law without appeal to classical electrodynamics. He divided the phase space of light quanta into cells of volume h^3 and counted the ways in which quanta could be distributed among them. Einstein translated the paper and extended its method to material particles, and the result became Bose–Einstein statistics. This paper re-reads Bose's derivation as an exercise in combinatorics. It shows that Bose's own counting, in terms of the numbers of cells that contain a given number of quanta, is exactly equivalent to the “stars and bars” count of indistinguishable objects in distinguishable boxes, and it proves the equivalence with a generating function. Exact enumerations confirm the equivalence for small systems and show how sharply it departs from the count for distinguishable quanta. Numerical maximisation of Bose's weight recovers the geometric distribution of quanta per cell that yields Planck's law, and exact finite-system calculations show the convergence of the single-cell distribution and of the fluctuations to their thermodynamic limits, including Einstein's fluctuation formula with its particle and wave terms. The paper relates Bose's counting to Planck's combinatorial argument of 1900, to Natanson's analysis of 1911 and to the Ehrenfest–Kamerlingh Onnes derivation of 1915, and argues that the novelty of Bose's work lay less in a new formula than in a new choice of the objects to be counted.

Keywords: S. N. Bose; Bose–Einstein statistics; Planck's law; combinatorics; indistinguishability; stars and bars; photon statistics; history of quantum theory

I. INTRODUCTION

Planck's law for the spectral energy density of black-body radiation,

$$u_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} d\nu$$

In 1924 Satyendra Nath Bose, a reader in physics at the University of Dacca, supplied one. His four-page paper derived Eq. (1) by treating radiation as a gas of light quanta and counting the ways in which quanta could be distributed among cells of phase space [6], [7]. Einstein translated it into German, arranged its publication, and extended the method to the ideal gas of material particles [8], [9]. The result is the statistics now named after both men [10]–[12]. Bose's work was one of the most important contributions to the physics of the period from India, whose early development of modern physics has also been surveyed in a Kannada-language doctoral analysis [13].

The question of what exactly was new in Bose's counting has been discussed by physicists and historians [14], [15]. This paper approaches it from the side of combinatorics. It asks precisely what Bose counted, proves that his count is equivalent to a standard count of indistinguishable objects,

checks the equivalence and its consequences numerically, and compares it with the counts used by Planck and his successors.

II. BOSE'S DERIVATION

Bose had taught the new physics at the University College of Science in Calcutta from 1916 and moved to Dacca in 1921 [10], [11]. In 1924 he sent his paper to Einstein after it had failed to find acceptance with a British journal, with a request that Einstein arrange its publication in German if he thought it worthwhile. Einstein did so, adding a note in which he described the derivation as an important advance and announced that he would apply the method to the quantum theory of the ideal gas [3], [6], [12].

Bose's argument can be summarised in four steps [6], [7]. First, he counted the cells. A quantum of frequency ν has momentum $h\nu/c$. The phase-space volume available to quanta with frequencies between ν and $\nu + d\nu$ in a volume V is $V \cdot 4\pi p^2 dp$, and dividing by h^3 and multiplying by 2 for the two states of polarisation gives the number of cells

$$A^s = \frac{8\pi\nu^2 V}{c^3} dv$$

for the frequency interval labelled s . This step reproduces the classical mode count without electrodynamics, and it was the feature Einstein singled out.

Second, he described a distribution of quanta by the numbers p_r 's of cells in interval s that contain exactly r quanta. The number of ways of realising a given set of numbers $\{p_r\}$ is

$$W = \prod_s \frac{A^s!}{p_0^s! p_1^s! p_2^s! \dots}$$

Third, he maximised $\ln W$ subject to the constraints

$$\sum_r p_r^s = A^s, \quad \sum_s \sum_r r p_r^s h\nu^s = E$$

Using Stirling's approximation and Lagrange multipliers, the most probable distribution is geometric in r :

$$p_r^s = A^s (1 - e^{-h\nu^s/k_B T}) e^{-r h\nu^s/k_B T}$$

Fourth, the mean number of quanta per cell follows as

$$\bar{n}^s = \frac{1}{A^s} \sum_r r p_r^s = \frac{1}{e^{h\nu^s/k_B T} - 1}$$

and multiplying by the number of cells, Eq. (2), and by the energy $h\nu$ of each quantum gives Planck's law, Eq. (1).

III. A COMBINATORIAL RE-READING

1. Occupation vectors and occupancy numbers

Consider a single frequency interval with A cells and N quanta. A microscopic configuration can be described in two ways. The first is an occupation vector $(n_1, n_2, \dots, n_A)$ that lists the number of quanta in each cell, with $\sum_j n_j = N$. The second is the set of occupancy numbers $\{p_r\}$ that Bose used, where p_r is the number of cells that contain r quanta. Each occupation vector determines one set of occupancy numbers, and each set of occupancy numbers corresponds to

$$\frac{A!}{\prod_r p_r!}$$

occupation vectors, namely the number of ways of assigning the values r to the A labelled cells. Bose's weight, Eq. (3), is therefore the number of occupation vectors with a given occupancy profile. In this count the cells are distinguishable, because they are labelled, but the quanta are not: an occupation vector says how many quanta a cell contains, not which ones.

2. The equivalence with stars and bars

Summing Eq. (7) over all occupancy profiles consistent with A cells and N quanta must give the total number of

occupation vectors. That total is the number of ways of placing N indistinguishable objects in A distinguishable boxes, which is given by the "stars and bars" argument: arrange N stars and $A - 1$ bars in a row, and read the number of stars between consecutive bars as the occupation of each box [16]. Hence

$$\sum_{\{p_r\}: \sum p_r = A, \sum r p_r = N} \frac{A!}{\prod_r p_r!} = \binom{N + A - 1}{N}$$

A short proof uses a generating function. The number of occupation vectors is the coefficient of t^N in

$$(1 + t + t^2 + \dots)^A = (1 - t)^{-A} = \sum_{N=0}^{\infty} \binom{N + A - 1}{N} t^N \quad (4)$$

and collecting the terms of the left-hand product according to how many factors contribute t^r reproduces the multinomial sum on the left of Eq. (8). Bose's occupancy-number description and the stars-and-bars description are two bookkeepings of the same set.

Fig. 1 illustrates the case $A = N = 3$. There are ten occupation vectors. They fall into three occupancy profiles: one cell with three quanta (three vectors), one cell with two and one with one (six vectors), and three cells with one each (one vector). The weights 3, 6 and 1 are the values of Eq. (7), and their sum is $\binom{5}{3} = 10$. Had the quanta been distinguishable, there would have been $3^3 = 27$ configurations.

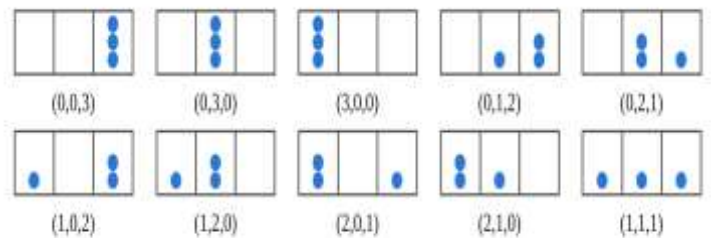


Fig. 1. The ten configurations of three indistinguishable quanta in three distinguishable cells, labelled by their occupation vectors.

3. What Bose did not count

The contrast with classical counting is clearest if the quanta are treated as distinguishable. The number of ways of distributing N labelled quanta among A cells with occupations n_j is the multinomial coefficient $N! / \prod_j n_j!$, and the total is A^N . Maximising this weight under the same energy constraint gives a Poisson distribution of quanta per cell with mean $\bar{n} = e^{-h\nu/k_B T}$, which leads to Wien's law [17] rather than Planck's. Classical equipartition of energy among

the modes of the field gives instead $\bar{n} = k_B T / h\nu$ and the Rayleigh–Jeans law [18], which agrees with Planck’s law only at low frequencies, where $\bar{n} \gg 1$. The three occupations are compared in Fig. 2. Bose’s counting differs from the classical one only in the treatment of the quanta as indistinguishable, but that difference turns Wien’s law into Planck’s. The difference is concentrated at low frequencies.

The maximum of the Planck spectrum lies at $x = h\nu/k_B T = 2.821$, that of the Wien form at $x = 3$, and for $x > 5$ the two differ by less than 1%. Wien’s law was accurate in the region where it was first tested precisely because, there, quanta seldom share a cell.

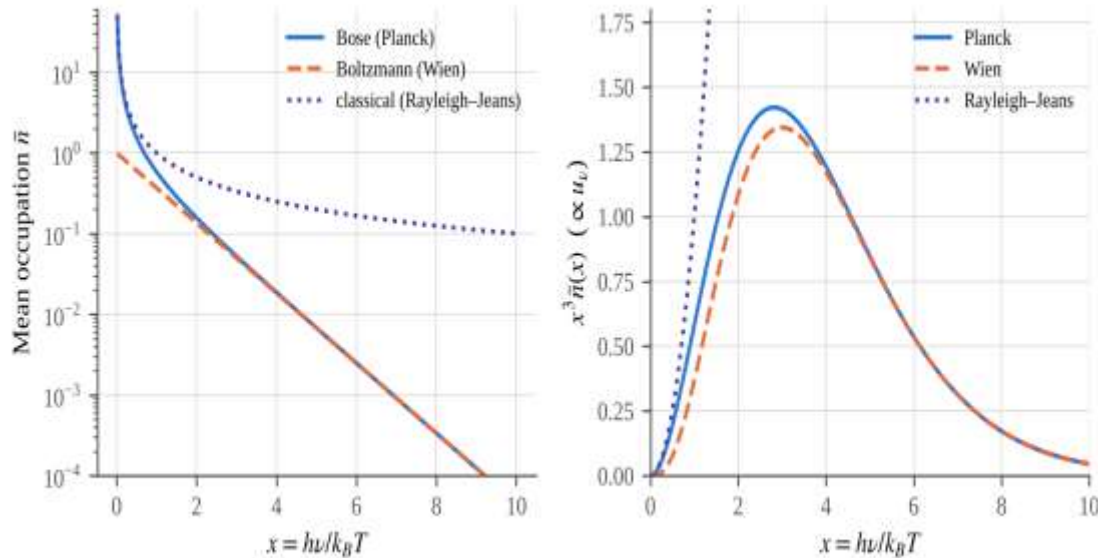


Fig. 2. Left: mean occupation per cell for Bose counting (Planck), distinguishable quanta (Wien) and classical equipartition (Rayleigh–Jeans). Right: the corresponding spectral energy densities, proportional to $x^3 \bar{n}(x)$ with $x = h\nu/k_B T$.

4. Planck, Natanson and Ehrenfest

The binomial coefficient of Eq. (8) is not new. Planck’s combinatorial argument of 1900 counted the number of ways of distributing P indivisible energy elements among N resonators,

$$W_{\text{Planck}} = \frac{(N + P - 1)!}{P! (N - 1)!}$$

a formula that has exactly the stars-and-bars form [1], [2]. Ehrenfest and Kamerlingh Onnes gave in 1915 a simple derivation of it by arranging energy elements and partitions in a row [19]. Natanson had already noted in 1911 that Planck’s count implies a statistical dependence between energy elements that is absent for independent particles [15], [20].

The difference between Planck and Bose lies in the objects. In Planck’s count the energy elements are distributed among material resonators; in Bose’s count the light quanta themselves are distributed among cells of their own phase space. The same formula thus acquires a new physical meaning: the light quanta are the particles of the gas, and their indistinguishability is a property of the gas rather than a device for counting energy. It was this reinterpretation that

allowed Einstein to apply the same reasoning to atoms [3], [8], [9].

5. Profiles, partitions and weights

The occupancy profiles themselves have a simple combinatorial identity. A profile $\{p_r\}$ records how many cells hold one quantum, how many hold two, and so on, without saying which cells they are. It is therefore an integer partition of N into at most A parts. For three quanta in three cells there are three partitions, 3, 2 + 1 and 1 + 1 + 1, which are the three groups in Fig. 1. For five cells and four quanta there are 5 profiles, for six cells and six quanta 11, and for eight cells and ten quanta 40, against 70, 462 and 19 448 occupation vectors respectively.

This distinction matters. If every profile were counted once, as if the cells too were indistinguishable, the statistics would be quite different, and it would not yield Planck’s law. Bose’s weight, the multinomial factor of Eq. (7), counts each profile as many times as there are ways of assigning it to labelled cells. In the language of modern statistical mechanics, Bose treated the single-particle states as distinguishable and the particles as indistinguishable, which is exactly the

prescription of Bose–Einstein statistics [21], [22]. The same prescription is implicit in Planck’s formula, Eq. (10), where the resonators play the role of labelled cells and the energy elements that of unlabelled quanta.

6. The variational step

For large numbers of cells, Stirling’s approximation $\ln m! \approx m \ln m - m$ turns the logarithm of Eq. (3) into a smooth function of the occupancy numbers. Introducing Lagrange multipliers λ^s for the number of cells in each interval and β for the total energy, the condition for a maximum is

$$\frac{\partial}{\partial p_r^s} \left[\ln W - \sum_s \lambda^s \sum_r p_r^s - \beta \sum_s \sum_r r p_r^s h\nu^s \right] = -\ln p_r^s - \lambda^s - \beta r h\nu^s = 0$$

so that $p_r^s \propto e^{-\beta r h\nu^s}$, the geometric form of Eq. (5). The multiplier β is identified with $1/k_B T$ by requiring that $\partial S / \partial E = 1/T$ with $S = k_B \ln W$. Only two kinds of constraint enter, the number of cells and the total energy. There is no constraint on the number of quanta, because light quanta are created and absorbed freely by matter.

7. Einstein’s extension to material particles

For a gas of atoms the number of particles is fixed, and a third constraint, $\sum_s \sum_r r p_r^s = N$, must be added with its own multiplier. Carrying this through gives Einstein’s distribution [8], [9]

$$\bar{n}(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/k_B T} - 1}$$

where μ is the chemical potential, fixed by the number of particles. For light quanta $\mu = 0$ and Eq. (12) reduces to Eq. (6). The combinatorial content is unchanged: atoms, like Bose’s quanta, are counted as indistinguishable objects in distinguishable cells. Einstein’s second paper drew from this distribution the prediction that, below a critical temperature, a finite fraction of the atoms must occupy the lowest state, the phenomenon now called Bose–Einstein condensation [3], [9]. The step from Bose’s photon gas to Einstein’s atomic gas required no new combinatorics, only an additional constraint.

IV. COMPUTATIONAL CHECKS

1. Exact enumeration

Table 1 compares, for small systems, the number of occupation vectors obtained by brute-force enumeration, the

sum of Bose’s weights over all occupancy profiles, the binomial coefficient of Eq. (8), and the number of configurations of distinguishable quanta. The first three agree exactly in every case, while the last grows far faster. Already for three cells and five quanta the distinguishable count is more than eleven times larger, and for eight cells and ten quanta there are 19 448 configurations of indistinguishable quanta but more than 10^9 of distinguishable ones.

Table 1. Exact counts of configurations for small systems. The brute-force count was omitted for the last row because of its size.

Cel ls A	Quan ta N	Brut e- forc e coun t	Sum of Bose weigh ts, Eq. (8)	$\binom{N+A-1}{N}$	Distinguishable, A^N
2	2	3	3	3	4
3	3	10	10	10	27
4	3	20	20	20	64
3	5	21	21	21	243
5	4	70	70	70	625
6	6	462	462	462	46 656
8	10	–	19 448	19 448	1 073 741 824

2. Maximising Bose’s weight

To check the third step of Bose’s argument without Stirling’s approximation for the final answer, $\ln W$ was maximised numerically for $A = 1000$ cells and $N = 1500$ quanta, treating the p_r as continuous variables and imposing the constraints of Eq. (4) by sequential quadratic programming. Fig. 3 compares the result with the geometric distribution of Eq. (5), in which $e^{-h\nu/k_B T} = \bar{n}/(1 + \bar{n})$. The two agree to within 6×10^{-5} for the first ten occupancies, and the fitted value $h\nu/k_B T = \ln[(1 + \bar{n})/\bar{n}] = 0.511$ follows directly. Occupancies up to $r = 39$ were included; the geometric distribution predicts fewer than 0.002 cells with more than 25 quanta, so the truncation has no visible effect. The calculation uses only the continuous form of $\ln W$ and the two constraints, and it therefore tests the logic of Bose’s variational step independently of the algebra that leads to Eq. (5).

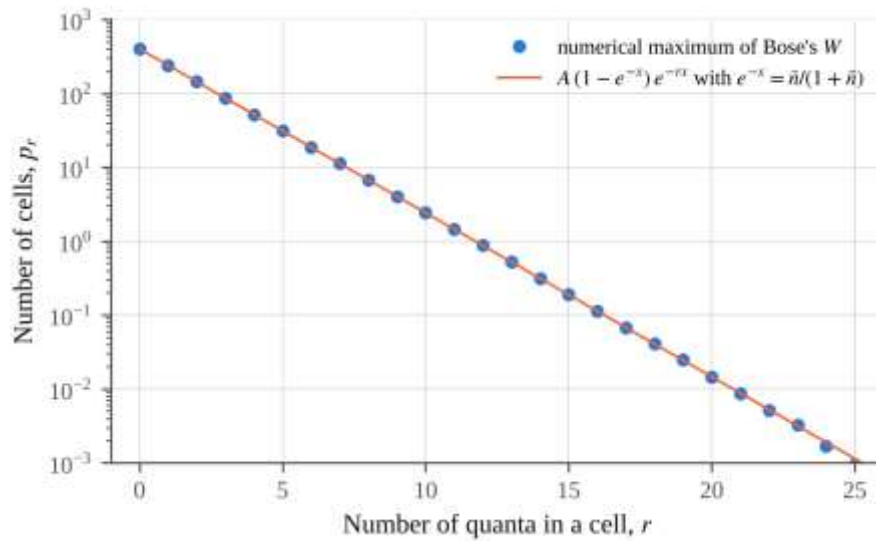


Fig. 3. Occupancy numbers p_r that maximise Bose's weight for $A = 1000$ cells and $N = 1500$ quanta (points), compared with the geometric distribution of Eq. (5) (line).

3. Entropy

With the most probable distribution, the entropy per cell takes the form

$$\frac{S}{Ak_B} = (1 + \bar{n})\ln(1 + \bar{n}) - \bar{n}\ln\bar{n}$$

for Bose counting, and $\bar{n}(1 - \ln\bar{n})$ for distinguishable quanta with the usual $N!$ correction [21], [22]. Fig. 4

compares them. They coincide when $\bar{n} \ll 1$, where quanta rarely share a cell and the distinction between counting methods disappears; this is the Wien region of the spectrum. They diverge when $\bar{n} \gtrsim 1$. The classical expression even becomes negative for $\bar{n} > e$, a sign that it cannot describe densely occupied cells. Equation (13) is also the large- A limit of $\ln\binom{N+A-1}{N}/A$; for $\bar{n} = 1.5$ the relative difference between the two is 19% for $A = 10$, 2.4% for $A = 100$, 0.30% for $A = 1000$ and 0.037% for $A = 10\,000$.

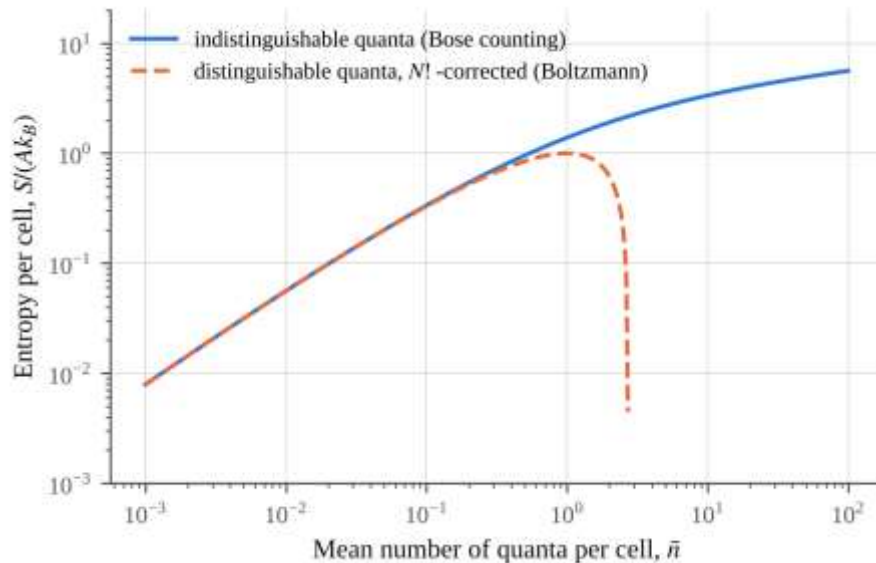


Fig. 4. Entropy per cell as a function of the mean number of quanta per cell for Bose counting and for distinguishable quanta.

4. The distribution of quanta in one cell

The combinatorial picture yields an exact result for a finite system. If N quanta are distributed among A cells with all occupation vectors equally likely, the probability that a given cell holds k quanta is the number of ways of distributing the remaining $N - k$ quanta among the other $A - 1$ cells, divided by the total:

$$P(k) = \frac{\binom{N-k+A-2}{N-k}}{\binom{N+A-1}{N}} \rightarrow \frac{\bar{n}^k}{(1+\bar{n})^{k+1}}$$

The limit, for $A \rightarrow \infty$ at fixed $\bar{n} = N/A$, is the geometric distribution, which is the single-cell form of Eq. (5). For

distinguishable quanta the corresponding distribution is binomial and tends to the Poisson distribution. Fig. 5 shows both for $\bar{n} = 2$. Even with only ten cells the Bose distribution is close to geometric, and the difference measured by the total-variation distance falls from 0.033 for $A = 10$ to 0.0030 for $A = 100$ and 0.00030 for $A = 1000$, decreasing in proportion to $1/A$. The rate is what one expects: placing k quanta in the chosen cell removes them from a reservoir of N quanta and $A - 1$ cells, and the resulting correction to the probabilities is of relative order k/N and $1/A$. The distinguishable case is markedly different: it makes high occupations far less likely, as quanta that are placed independently rarely cluster.

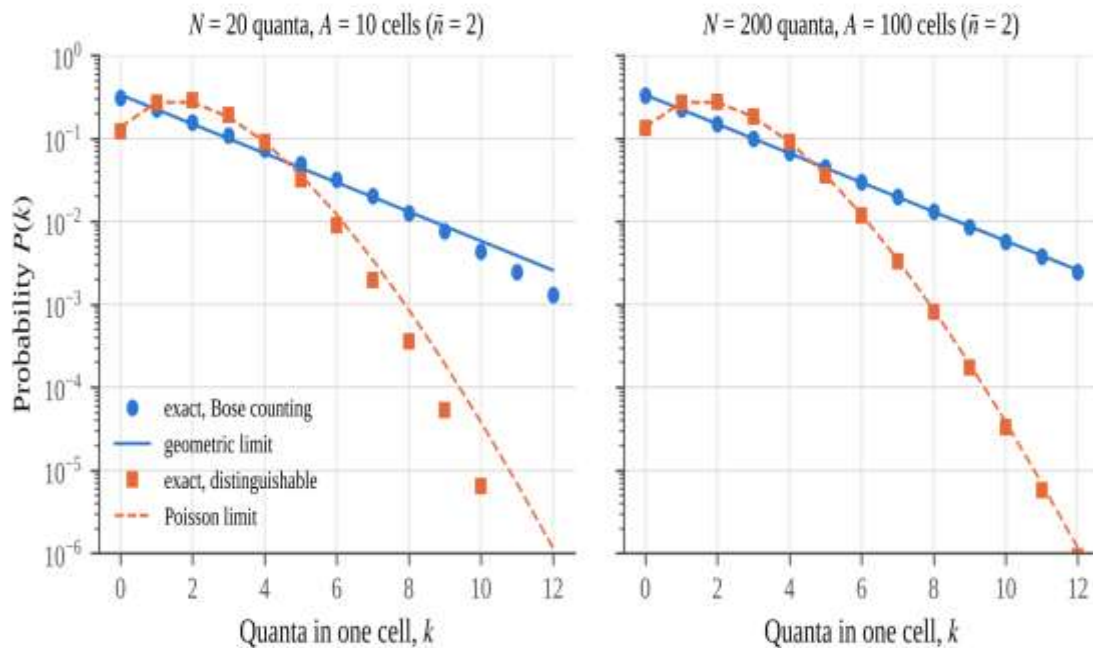


Fig. 5. Probability that one cell holds k quanta: exact finite-system results for Bose counting and for distinguishable quanta, with their geometric and Poisson limits, for $\bar{n} = 2$.

5. Fluctuations

The variance of the geometric distribution is

$$\langle (\Delta n)^2 \rangle = \bar{n} + \bar{n}^2$$

which is Einstein's 1909 result for the energy fluctuations of radiation, expressed per cell [5], [23]. The first term is the fluctuation expected for independent particles; the

second is the fluctuation of interfering waves. In Bose's counting both arise from a single cause: the indistinguishability of quanta makes clustering more likely than independent placement would. Fig. 6 shows Eq. (15) together with the exact variances from Eq. (14) for 400 cells. For $\bar{n} = 2$ the exact variance is 4.91 for 10 cells, 5.88 for 100 cells and 5.99 for 1000 cells, approaching the limit of 6.

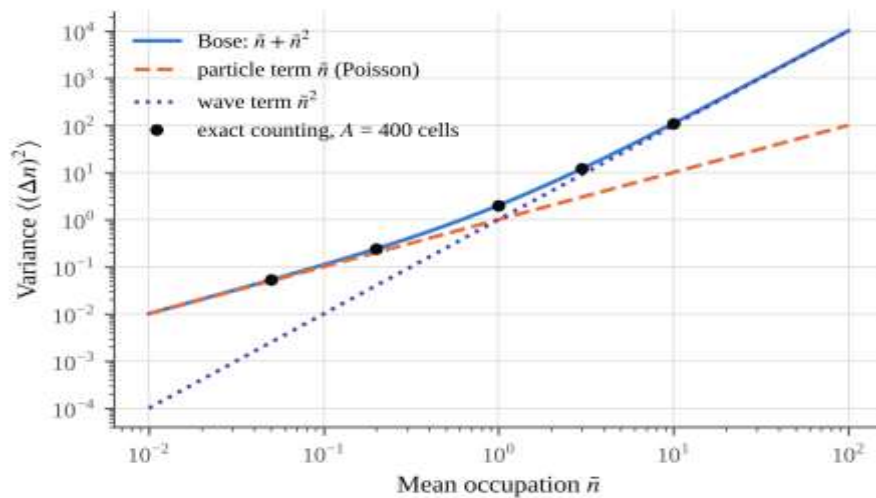


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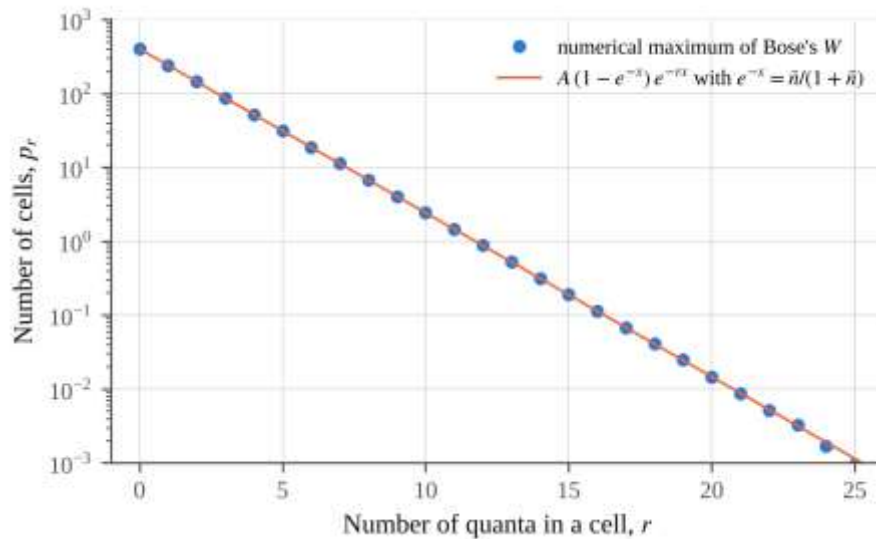


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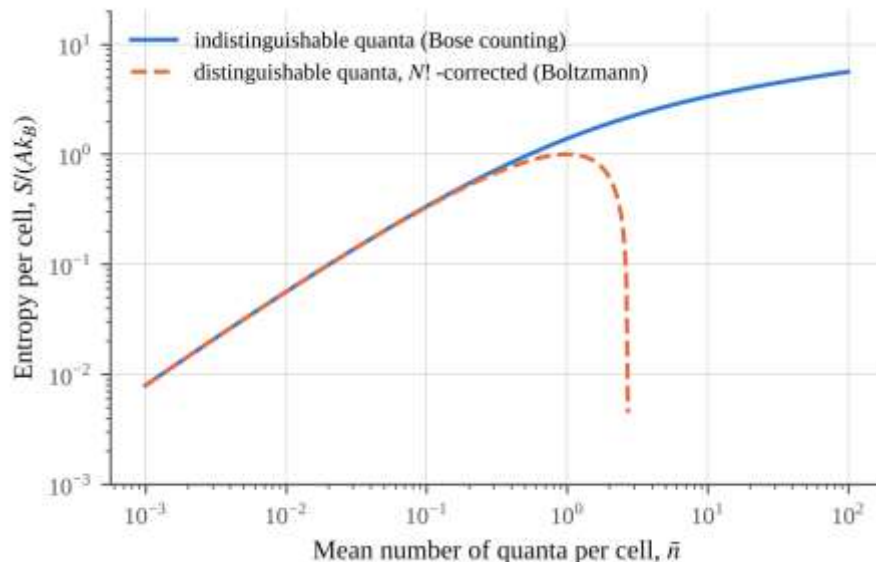


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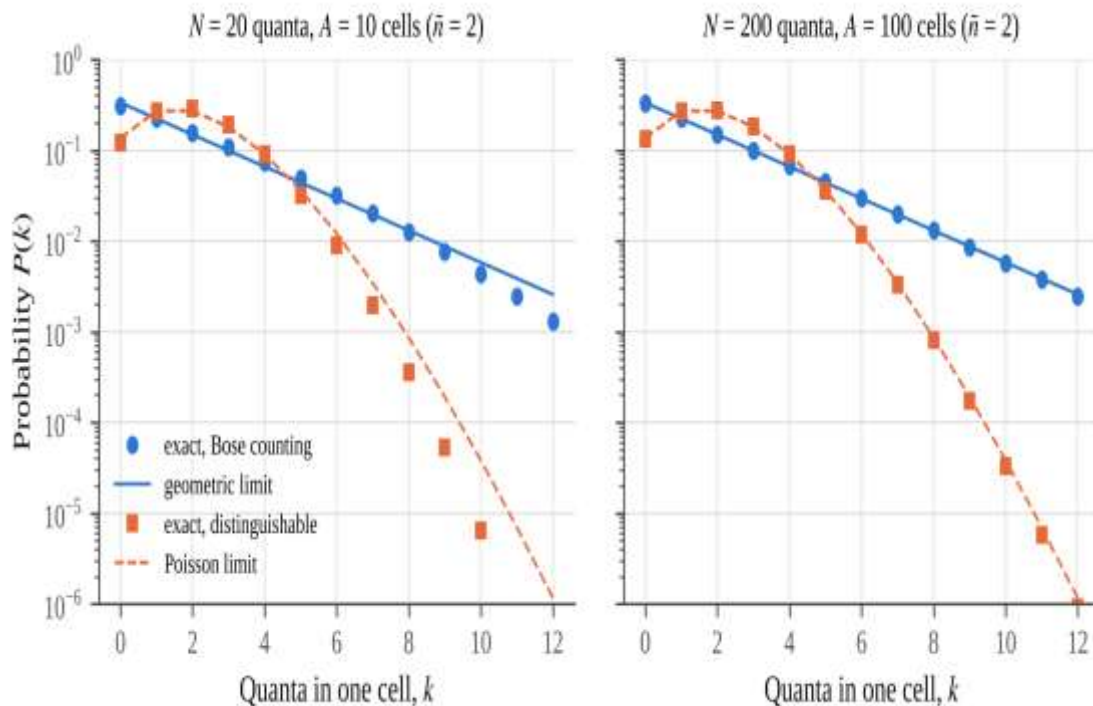


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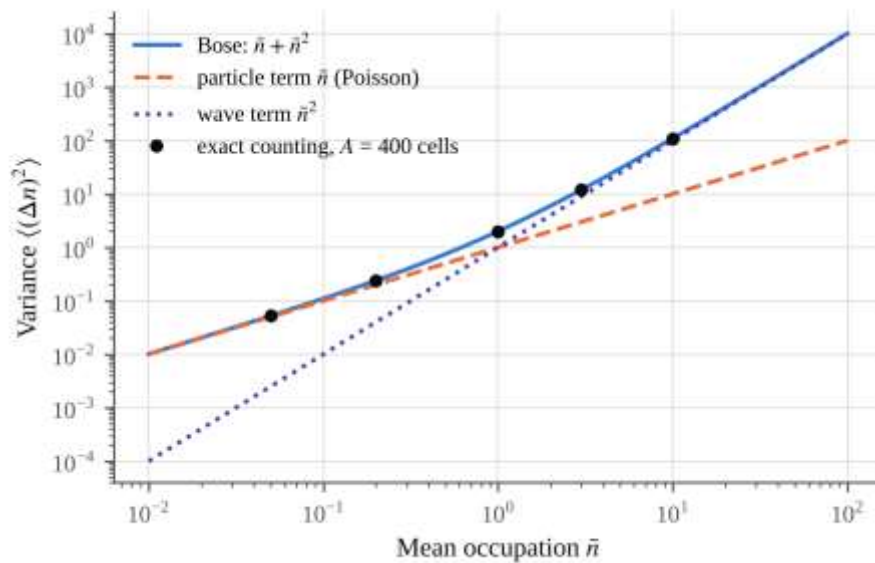


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VI. DISCUSSION

The re-reading clarifies what was and was not new in Bose's paper. The binomial coefficient at the heart of the count had been used by Planck in 1900 and explained by Ehrenfest and Kamerlingh Onnes in 1915. What was new was its application to light quanta in phase-space cells. Bose took the particle picture of radiation seriously enough to count quanta directly, and he counted them in a way that did not distinguish one quantum from another. The cell count of Eq. (2), which replaced Planck's electrodynamic mode count, made the derivation independent of classical theory.

Whether Bose recognised that he was introducing a new kind of statistics has been debated. His paper does not discuss distinguishability explicitly, and his counting is expressed through occupancy numbers that make the equivalence with stars and bars less obvious than it is in hindsight. Delbrück asked whether the new statistics was reached by serendipity [14], and Darrigol traced the long history of combinatorial arguments in which indistinguishability was implicit before it was recognised [15]. The analysis of Section 3 supports a middle view. Bose's weight, Eq. (3), is not an accident. It is the correct count of occupation vectors with labelled cells and unlabelled quanta, and it is the natural count once one decides that a configuration is specified by how many quanta each cell contains. The decisive step was that decision. Bose's biographers emphasise his independence of mind and his habit of approaching problems from first principles, which are consistent with such a choice [10]–[12].

The same combinatorics now has a direct experimental meaning. In quantum optics the number of photons counted in a single mode of thermal light follows the geometric distribution of Eq. (14), whereas light from a well-stabilised laser follows the Poisson distribution of independent particles [23]. The excess variance $\bar{n}^2$ of Eq. (15) appears as the bunching of photons from thermal sources, which can be measured with photon-counting detectors. Bose's way of counting light quanta is therefore not only a derivation of Planck's law but a description of a measurable property of light.

The episode also belongs to the history of physics in India. Bose and his contemporary Meghnad Saha were among the first lecturers in the postgraduate science departments of the University of Calcutta, where they were required to teach thermodynamics, relativity and the quantum theory to advanced students while learning these subjects themselves from the journals [11], [13]. Bose's derivation, written after his move to Dacca, grew out of that experience. It is an example of how teaching the foundations of a subject can lead to a fresh look at them, and of how a scientist working far from the established centres could contribute to their central problems.

The combinatorial view also has pedagogical value. Students meet Bose–Einstein statistics in textbooks through grand-canonical arguments that hide the counting [21], [22]. The exact finite-system results of Section 4, which require nothing more than binomial coefficients, show directly how indistinguishability changes the distribution of quanta, why it increases fluctuations, and how the thermodynamic limits are approached. They provide an accessible route to one of

the central ideas of quantum physics, and they connect it with a contribution made in an Indian university a century ago. Two limitations of the present analysis should be noted. It treats a single frequency interval at a time and assumes that all occupation vectors are equally probable, which is the fundamental postulate of statistical mechanics applied to an isolated system; the coupling between intervals enters only through the shared temperature. And it reconstructs Bose's reasoning from his published paper and its translation, without access to drafts or correspondence that might show how he arrived at it.

VII. CONCLUSION

Bose's 1924 derivation of Planck's law rests on a combinatorial choice: light quanta are counted as indistinguishable objects distributed among distinguishable cells of phase space. His occupancy-number weight is exactly equivalent to the stars-and-bars count, as a generating-function argument and exact enumeration confirm. Maximising it gives the geometric distribution of quanta per cell and hence Planck's law, and exact finite-system calculations show how the single-cell distribution and Einstein's fluctuation formula emerge from the counting. The formula Bose used was old; the objects he counted were new. That change of objects turned a device for counting energy into the statistics of a gas of quanta, and it was the foundation on which Einstein built Bose–Einstein statistics. A century later, the same simple count still describes the statistics of thermal light and of every gas of identical bosons.

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