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Fractional Differential Equations (FDEs) in Viscoelasticity or Anomalous Diffusion

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Abstract- — Classical diffusion models based on Fick's law assume Brownian motion and local transport, leading to a mean squared displacement that grows linearly with time. However, many physical, biological, and engineering systems exhibit anomalous diffusion, where the mean squared displacement follows a power law in time rather than a linear relationship. Such behavior commonly arises in heterogeneous porous materials, crowded biological environments, polymeric systems, and disordered media, where long trapping times and memory effects invalidate standard integer-order diffusion equations. Despite significant progress in fractional modeling, there remains a need for mathematically consistent and computationally efficient formulations that clearly link the physical origin of anomalous transport to robust numerical implementation. In this paper, we develop a time-fractional diffusion model using the Caputo fractional derivative to represent memory-dependent transport induced by heavy-tailed waiting times. Starting from the conservation of mass and a constitutive relation with temporal memory, we derive a physically meaningful fractional diffusion equation. An analytical solution for a benchmark initial-boundary value problem is presented using Laplace and Fourier transforms, and a numerical approximation based on the L1 finite difference scheme is constructed. The stability and convergence properties of the numerical method are discussed. Numerical experiments demonstrate that the fractional order controls the transition from normal to subdiffusive transport and accurately reproduces power-law mean squared displacement behavior. The model captures anomalous transport with significantly fewer parameters than multi-scale classical alternatives. These results show that fractional differential equations provide an effective and parsimonious framework for describing memory-driven diffusion processes, with direct relevance to transport in porous media, biological tissues, and complex soft matter systems.

Keywords— Fractional differential equations; Anomalous diffusion; Caputo fractional derivative; Time-fractional diffusion equation; Viscoelasticity modelling; Memory effects; Sub diffusion processes; Fractional calculus; Mean squared displacement; Non-local transport phenomena; L1 finite difference scheme; Stability and convergence analysis.

I. INTRODUCTION

Diffusion is one of the most fundamental transport mechanisms in physics, chemistry, biology, and engineering. In the classical theory, particle motion is modeled as a Markovian random process, and the resulting mean squared displacement grows linearly in time. In one spatial dimension, this behavior is expressed as

$$\langle x^{2(t)} \rangle \propto t.$$

This relationship is characteristic of normal diffusion and follows directly from Fick's law and the standard diffusion equation. However, a large number of experimental studies have shown that many real systems do not obey this linear scaling. Instead, the mean squared displacement often follows the generalized law

$$\langle x^{2(t)} \rangle \propto t^{\alpha}, \quad 0 < \alpha < 1,$$

which is referred to as subdiffusion, while the case $\alpha > 1$ corresponds to superdiffusion. Subdiffusive transport is especially common in crowded intracellular media, porous geological formations, amorphous semiconductors, hydrogels, and polymer networks. In such materials, particles may experience trapping, sticking, crowding, or obstruction, leading to strong memory and nonlocal effects.

Classical integer-order differential equations are often inadequate for modeling such transport. Their local-in-time structure cannot naturally represent long-time memory or heavy-tailed waiting-time distributions. As a result, standard diffusion equations tend to overpredict spreading rates and fail to reproduce experimentally observed power-law scaling. One possible remedy is to introduce many empirical fitting parameters, but this often reduces interpretability and physical transparency.

Fractional differential equations (FDEs) have emerged as a powerful alternative for modeling anomalous diffusion. These equations replace the classical first-order time derivative with a derivative of non-integer order, thereby introducing a mathematically precise form of memory. The physical basis for such models is well established through the theory of Continuous-Time Random Walks (CTRWs), where heavy-tailed waiting-time distributions lead, in the macroscopic limit, to time-fractional diffusion equations. Fractional models also naturally produce Mittag-Leffler relaxation and power-law transport behavior, both of which are widely observed in experiments. The foundations of fractional calculus in physical modeling were developed in important works by Podlubny, Mainardi, Metzler, and Klafter, among others. Mainardi established key links between viscoelasticity and fractional operators, while Metzler and Klafter provided a comprehensive framework connecting anomalous transport to stochastic processes and fractional kinetics. More recent studies have focused on numerical approximation, inverse parameter estimation, variable-order models, and machine learning approaches such as fractional physics-informed neural networks. Nevertheless, there remains an important gap between physically derived fractional models and computationally practical schemes that can be readily implemented and validated. The objective of this paper is to develop and analyze a time-fractional diffusion model for anomalous transport in heterogeneous media. We derive the governing equation from mass conservation and a constitutive flux law with memory, adopt the Caputo fractional derivative to preserve physically meaningful initial conditions, and present both analytical and numerical solution approaches. A finite difference method based on the L1 discretization is introduced, and its consistency and stability are discussed. Numerical experiments are then used to investigate the effect of the fractional order on transport dynamics and to illustrate the transition from normal to subdiffusion.

The remainder of this paper is organized as follows. Section 2 introduces the mathematical preliminaries required for the analysis, including the Caputo derivative and the Mittag-Leffler function. Section 3 presents the derivation of the fractional diffusion model. Section 4 describes the analytical and numerical methodology. Section 5 contains numerical results and physical discussion. Finally, Section 6 summarizes the main findings and outlines possible directions for future research.

II. MATHEMATICAL PRELIMINARIES

In this section, we briefly introduce the key mathematical tools used in the development of the model.

1. Caputo Fractional Derivative

For a sufficiently smooth function $(u(t))$, the Caputo fractional derivative of order (α), where $(0 < \alpha < 1)$, is defined by

$$ {}^{C}D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, u'(\tau)\, d\tau, \quad 0 < \alpha < 1. $$

Here, (Γ) denotes the Gamma function. The Caputo derivative is particularly useful in physical applications because it allows the use of standard initial conditions such as

$$ u(x,0) = u_{0(x)}, $$

which is generally more natural than the fractional initial conditions associated with the Riemann-Liouville derivative.

2. Laplace Transform of the Caputo Derivative

The Laplace transform of the Caputo derivative plays a central role in the analytical solution of fractional differential equations. If

$$ \mathcal{L}\{u(t)\}(s) = U(s), $$

then, for $0 < \alpha < 1$,

$$ \mathcal{L}\{{}^{C}D_t^{\alpha} u(t)\}(s) = s^{\alpha} U(s) - s^{\alpha-1} u(0). $$

]This formula closely resembles the classical Laplace transform of the first derivative and is one reason why the Caputo derivative is favored in applied problems.

3. Mittag-Leffler Function

The Mittag-Leffler function is the fractional analogue of the exponential function and appears naturally in the solution of linear fractional differential equations. It is defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. $$

A more general two-parameter form is

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0. $$

When $(\alpha = 1)$, the Mittag-Leffler function reduces to the exponential function:

$$ E_{1(z)} = e^{z}. $$

This function governs the temporal relaxation of many fractional systems and replaces the exponential decay of classical diffusion.

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3. Formulation of the Physical Model

1. Conservation of Mass

Let $u(x,t)$ denote the concentration of diffusing particles at position x and time t. The local conservation of mass in one dimension is given by

$$\frac{\partial u}{\partial t}+\frac{\partial J}{\partial x}=0.$$

where $J(x,t)$ is the particle flux.

For normal diffusion, Fick's law states

$$J(x,t)=-D\,\frac{\partial u}{\partial x}.$$

where D is the diffusion coefficient. Substituting this into the conservation law yields the classical diffusion equation

$$\frac{\partial u}{\partial t}=D\,\frac{\partial^2 u}{\partial x^2}.$$

This model assumes that the flux responds instantaneously to the concentration gradient. In complex media, however, the transport process has memory, and the flux may depend on the entire history of the concentration gradient.

2. Constitutive Flux Law with Memory

To incorporate memory effects, we introduce a nonlocal constitutive relation of the form

$$J(x,t)=-D\int_0^t K(t-\tau)\,\frac{\partial u(x,\tau)}{\partial x}\,d\tau.$$

where $(K(t))$ is a memory kernel. For anomalous subdiffusion, a power-law kernel is appropriate:

$$K(t)=\frac{t^{-\alpha}}{\Gamma(1-\alpha)},\quad 0<\alpha<1.$$

This kernel reflects long-time memory and is consistent with heavy-tailed waiting-time distributions in CTRW theory.

Substituting the constitutive law into the conservation equation gives

$$\frac{\partial u}{\partial t}=D\,\frac{\partial}{\partial x}\int_0^t K(t-\tau)\,\frac{\partial u(x,\tau)}{\partial x}\,d$$

Using the definition of the Caputo derivative, this can be recast into the time-fractional diffusion equation

$${}^{C}D_t^{\alpha}u(x,t)=D\,\frac{\partial^2 u(x,t)}{\partial x^2},\quad 0<\alpha<1.$$

This equation is the central model studied in this paper.

3. Physical Interpretation

The parameter α quantifies the degree of memory in the system. When $\alpha = 1$, the Caputo derivative reduces to the classical first-order derivative, and the model becomes the standard diffusion equation. When $0 < \alpha < 1$, the system exhibits sub diffusive behaviour due to long waiting times, trapping phenomena, or structural heterogeneity.

The corresponding mean squared displacement scales as

$$\langle x^{2(t)}\rangle \sim t^{\alpha},$$

which matches the experimentally observed law for anomalous diffusion.

4. Initial and Boundary Conditions

To define a well-posed problem, we consider the one-dimensional domain

$0 < x < L, \quad t > 0,$

with the initial condition

$$u(x,0) = f(x),$$

and homogeneous Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0.$$

These boundary conditions correspond physically to perfectly absorbing endpoints. Other conditions such as Neumann or Robin conditions could also be used depending on the application.

IV. METHODOLOGY

1. Analytical Solution by Separation of Variables

Consider the time-fractional diffusion equation

$${}^{C}D_t^{\alpha}u(x,t)=D\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

For $0 < x < L$, $t > 0$, with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0,$$

and initial condition

$$u(x,0) = f(x).$$

Assume a separable solution of the form

$$u(x,t) = X(x)T(t).$$

Substitution into the governing equation yields

$$X(x)\,{}^{C}D_t^{\alpha}T(t) = D\,T(t)\,X''(x)$$

Applying next step, we obtain

$$D_t^{\alpha}T(t)/D\,T(t)=\frac{X''(x)}{X(x)}$$

Thus, the spatial problem becomes

$$X''(x) + \lambda X(x) = 0,$$

with

$$X(0) = 0, \quad X(L) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi x}{L}\right), n=1,2,3,$$

The temporal part satisfies

$${}^{C}D_t^{\alpha}T_n(t)+D\,\lambda_n T_n(t)=0.$$

Its solution is given by the Mittag-Leffler function:

$$T_n(t) = E_{\alpha}(-D\,\lambda_n\,t^{\alpha}).$$

Therefore, the full solution is

$$u(x,t)=\sum_{n=1}^{\infty} b_n\,E_{\alpha}D\left(\frac{n\pi}{L}\right)^2 t^{\alpha}\sin\left(\frac{n\pi x}{L}\right).$$

where the Fourier coefficients are

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

This representation generalizes the classical exponential solution to the fractional setting.

2. Numerical Approximation: L1 Finite Difference Scheme

Analytical solutions are only available for limited classes of problems. For general domains or nonlinear extensions, numerical methods are necessary.

Let the temporal grid be defined by

$$t_n = \mathrm{n}\,\Delta t, \quad n = 0,1,\dots,\mathrm{N},$$

and the spatial grid by

$$x_i = \mathrm{i}\,\Delta x, \quad i = 0,1,\dots,\mathrm{M}.$$

Denote the numerical approximation by

$$u_i^n \approx u(x_i, t_n)$$

The L1 approximation of the Caputo derivative at time t_n is

$${}^C D_t^\alpha u(x_i, t_n) \approx \frac{1}{\Gamma(2-\alpha)(\Delta t)^\alpha}\sum_{k=0}^{n-1} a_k\left(u_i^{n-k} - u_i^{n-k-1}\right),$$

where

$$a_k = (k+1)^{\{1-\alpha\}} - k^{\{1-\alpha\}}.$$

The second derivative in space is approximated using the standard central difference formula:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx u_{i+1}^n - 2u_i^n + u_{i-1}^n(\Delta x)^2.$$

Combining these expressions yields the fully discrete scheme

$$\frac{1}{\Gamma(2-\alpha)(\Delta t)^\alpha}\sum_{k=0}^{\mathrm{n}-1} a_k\left(\mathrm{u}_\mathrm{i}^{\mathrm{n}-\mathrm{k}} - \mathrm{u}_\mathrm{i}^{\mathrm{n}-\mathrm{k}-1}\right) = \mathrm{D}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

This scheme is widely used because of its simplicity and consistency with the Caputo derivative.

3. Stability and Convergence

For sufficiently smooth solutions, the L1 scheme has temporal accuracy of order

$$O\left((\Delta t)^{\{2-\alpha\}}\right),$$

and the central difference approximation has spatial accuracy of order

$$O((\Delta x)^2).$$

Hence, the total truncation error is

$$O\left((\Delta t)^{\{2-\alpha\}} + (\Delta x)^2\right)$$

Under standard assumptions, the scheme is stable and convergent. In practice, the nonlocal memory term requires storage of all previous time levels, which increases the computational cost for long-time simulations. This is one of the main numerical challenges in fractional diffusion problems.

V. RESULTS AND DISCUSSION

1. Verification Against Analytical Solution

To verify the numerical implementation, we consider a benchmark problem with initial condition

$$\mathrm{f(x)} = \sin!\left(\frac{\pi \mathrm{x}}{\mathrm{L}}\right)$$

In this case, the exact solution is

$$\mathrm{u(x,t)} = \mathrm{E}_\alpha - \mathrm{D}\left(\frac{\pi}{\mathrm{L}}\right)^2 \mathrm{t}^\alpha \sin!\left(\frac{\pi \mathrm{x}}{\mathrm{L}}\right).$$

This provides a direct reference for computing the numerical error. The discrete (L^2) −error at time (t_n)may be defined as

$$E^n = \left(\Delta x\sum_{i=1}^{M-1}|u(x_i,t_n) - u_i^n|^2\right)^{\frac{1}{2}}.$$

A log-log plot of the error versus (Δt) confirms the expected convergence rate ($2-\alpha$) in time.

2. Effect of the Fractional Order

A central advantage of the fractional model is that the order α has a clear physical meaning. To study its effect, simulations are performed for several values such as

$$\alpha = 0.25,;\ 0.50,;\ 0.75,;\ 1.0.$$

When ($\alpha = 1$), the model reduces to classical diffusion and the concentration profile spreads rapidly. As (α) decreases, the spreading becomes slower, reflecting stronger trapping and memory effects. The concentration remains more localized for longer times, which is characteristic of subdiffusive transport.

This transition can also be quantified through the mean squared displacement, which follows

$$\left\langle x^{2(t)}\right\rangle \sim t^\alpha.$$

On a log-log plot of $\left(\left\langle x^{\{2(t)\}}\right\}\right)$ versus (t), the slope equals (α). Thus, the fractional order directly controls the transport exponent and provides an interpretable measure of anomaly.

3. Comparison with Classical Diffusion

The classical diffusion equation predicts an exponential decay of Fourier modes:

$$u_{n(t)} \sim e^{\{-D\lambda_n t\}}.$$

In contrast, the fractional model predicts Mittag-Leffler decay:

$$u_n(t) \sim E_\alpha(-D\,\lambda_n t^\alpha).$$

his decay is slower than exponential and exhibits long-time algebraic behavior. As a result, the fractional model is better suited for systems where diffusion persists anomalously over long time scales. Moreover, classical multi-exponential models often require many fitting parameters to reproduce long-tail behaviour, whereas the fractional model captures the same effect through the single order parameter α, together with the diffusion coefficient D . This parameter efficiency is a major practical advantage.

4. Relevance to Experimental Systems

The present model is directly relevant to several experimentally observed systems. In biological cells, tracer particles often exhibit subdiffusion due to crowding and binding events. In porous rocks, contaminant transport may be slowed by dead-end pores and adsorption. In polymer gels and hydrogels, molecular diffusion can be strongly hindered by the microstructure of the network.

In such systems, a classical diffusion model may substantially overestimate transport rates, while the time-fractional diffusion equation reproduces the experimentally observed power-law scaling using only a small number of physically meaningful parameters. This makes the model attractive for inverse identification and predictive simulation.

5. Limitations

Although the time-fractional model captures memory effects effectively, it also has limitations. First, the memory term increases computational cost because the full time history must be retained. Second, the constant-order model assumes a uniform memory effect throughout the process, which may not be appropriate in systems with evolving microstructure. Third, some experimental systems may require a combination of temporal and spatial fractional operators to capture both trapping and long-range jumps. These limitations motivate future development of fast algorithms, variable-order formulations, and data-driven hybrid approaches.

VI. CONCLUSION AND FUTURE DIRECTIONS

In this paper, a physically consistent and mathematically tractable model for anomalous diffusion in heterogeneous media has been developed using fractional differential equations. Starting from the conservation of mass and a constitutive relation with memory, we derived a time-fractional diffusion equation involving the Caputo derivative. This formulation naturally captures sub diffusive behavior caused by trapping, crowding, and long-time memory effects. An analytical solution for a benchmark problem was obtained in terms of the Mittag-Leffler function, demonstrating the fractional generalization of classical exponential relaxation. A numerical approximation based on the L1 finite difference scheme was also presented, and its accuracy and convergence properties were discussed. The results show that the fractional order α plays a fundamental role in controlling transport dynamics. As α decreases from 1 to 0, the system transitions from classical diffusion to increasingly subdiffusive behavior. The model reproduces the observed power-law scaling of the mean squared displacement and offers a parsimonious alternative to classical multi-parameter fitting models. From a physical standpoint, the proposed framework is relevant to transport in porous media, polymeric materials, hydrogels, biological tissues, and crowded intracellular environments. Its main advantage lies in its ability to encode memory effects through a small number of interpretable parameters. Future work may focus on several directions. First, fast memory-saving numerical methods should be developed to reduce computational complexity. Second, variable-order fractional models may better capture evolving transport mechanisms. Third, coupling diffusion with mechanics or reaction kinetics could provide a more realistic description of biological and soft-matter systems. Finally, integration with machine learning frameworks such as fractional physics-informed neural networks may open new possibilities for solving inverse and data-assimilation problems in complex media.

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