International Journal of Scientific Research & Engineering Trends
Volume 12, Issue 3, May-Jun-2026, ISSN (Online): 2395-566X

A Functional Analytic Framework for the Modeling of Fatigue and Legal Liability Allocation

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Abstract- This paper introduces a formal framework utilizing mathematical functional analysis to bridge the gap between empirical sleep science and jurisprudence. By treating fatigue trajectories as functions within infinite-dimensional Banach spaces, we formalize how biomathematical fatigue inputs intersect with duty-of-care allocations within tort and regulatory systems, shifting the legal focus from rigid shift-hour compliance to systemic accountability.

Keywords- Fatigue Risk Management; Sleep Science; Functional Analysis; Banach Spaces; Biomathematical Fatigue Modeling; Duty of Care; Tort Law; Regulatory Compliance; Systemic Accountability; Occupational Fatigue; Legal Frameworks; Risk Assessment; Workplace Safety; Jurisprudence; Human Factors.

I. INTRODUCTION

The quantification of legal liability traditionally rests on establishing a causal, predictable link between a human agent's state and a subsequent breach of duty. Occupational fatigue, or tiredness, degrades cognitive and psychomotor capacities, making it a critical variable across high-risk industries, including maritime transport, long-haul trucking, and healthcare (Gander et al., 2011; Sprajcer et al., 2022). Despite its severity, traditional legal frameworks evaluate liability through binary or ad hoc compliance metrics, such as rigid, static shift-hour limitations (Dawson et al., 2021). These regulatory frameworks fail to capture the dynamic, cumulative, and highly non-linear nature of cognitive decay caused by sleep restriction and circadian disruption (Akerstedt et al., 2014).

To bridge the gap between empirical sleep science and jurisprudence, this paper introduces a formal framework utilizing mathematical functional analysis. Functional analysis allows us to map the continuous state space of human fatigue onto a continuous topological space of legal liability. By treating fatigue trajectories as functions within infinite-dimensional Banach spaces, we formalize how biomathematical fatigue inputs intersect with the "balancing of interests" and duty-of-care allocations within the tort and regulatory systems (Benito-Ostolaza et al., 2021; Honn & Dinges, 2011).

II. MATHEMATICAL FORMALIZATION OF FATIGUE SPACES

To construct a rigorous mapping of fatigue, we must define the domain of human physiological states over a continuous time interval $I = [0, T]$. Let $f(t)$ represent an agent's instantaneous fatigue level at time t. Instead of examining isolated, discrete values, we conceptualize the entire fatigue history of an agent as a function belonging to a specific functional space.

We define the Fatigue Space F as a subset of the Banach space $L^p(I)$, equipped with the
standard norm:

$$\|f\|_p = \left(\int_0^T |f(t)|^p \, dt \right)^{1/p} \quad (1)$$

The choice of the index p determines how the legal model weights sustained moderate tiredness versus acute spikes in sleepiness. For instance, in an $L^2(I)$ Hilbert space structure, the inner product allows for the projection of a driver's or clinician's true fatigue profile against an idealized, safe performance baseline (Benito-Ostolaza et al., 2021).

Biomathematical models of fatigue (BMMs) act as operators that map shift scheduling histories, circadian phase variables, and prior sleep opportunities into this functional space F (A˚ kerstedt et al., 2014; Dawson et al., 2011). The accumulation of fatigue is highly non-linear, exhibiting path-dependency where the ordering and density of effortful tasks induce a compounding degradation of decision-making capacity (Jaydarifard et al., 2023; Williamson et al., 2011).

Furthermore, because sleep restriction yields cumulative neurobehavioral deficits, the functional norm captures the total physiological debt over time far better than prescriptive, hour-counting rules can manage (Van Dongen et al., 2003).

III. THE LIABILITY OPERATOR AND FUNCTIONAL MAPPING

The legal system acts as a mechanism that balances competing dimensions, such as operational efficiency, individual autonomy, and public safety (Benito-Ostolaza et al., 2021). In functional analytic terms, we formalize the determination of legal liability as an operator L that maps elements from the fatigue function space F into a multi-dimensional topological space of legal liability M.

Let u ∈ [0, 1] represent a normalized legal index where u = 0 signifies absolute compliance (no liability) and u = 1 signifies strict liability or a total breach of the standard of care. This approach expands upon static legal balancing models by incorporating a time-dependent continuity constraint (Caldwell et al., 2009). The outcome function u(t) changes continuously based on structural input criteria, which we denote as legal parameters (Benito-Ostolaza et al., 2021).

We define the Liability Optimization Functional as:

$$\mathcal{L}(f) = \int_0^T \omega(t) \cdot [f(t) - f_{\text{thresh}}(t)]_+ \, dt + \Phi(P) \tag{2}$$

- ω(t) is a continuous weighting function reflecting the contextual hazard level of the task at time t. For example, operating a commercial vehicle during a circadian trough represents a significantly higher hazard than resting in a stationary berth (Folkard & Tucker, 2003; Goode, 2003).
- fthresh(t) represents the legally permissible fatigue threshold at time t, derived from stan-dardized biomathematical limits and sleep medicine benchmarks (Belleville et al., 2014).
- [·]+ is the positive part operator, ensuring that only fatigue exceeding the acceptable physiological threshold contributes to liability accumulation.
- Φ(P) is a functional reflecting systemic legal parameters P , such as organizational safety culture, the presence of an active Fatigue Risk Management System (FRMS), or the commercial pressures exerted on the agent (Gander et al., 2011; Jaydarifard et al., 2023; Sprajcer et al., 2022).

By structuring L as a continuous linear operator on a compact subset of F, we ensure that small perturbations in an employee's fatigue profile do not yield wildly discontinuous jumps in liability assessment. This mathematical structure establishes a predictable, objective framework for tort and regulatory adjudication (Sallinen et al., 2008).

IV. LEGAL IMPLICATIONS: SHIFTING FROM AD HOC COMPLIANCE TO SYSTEMIC

Accountability

Traditional statutory frameworks rely on strict cutoff criteria, such as a commercial vehicle driver exceeding a 12-hour shift limit. However, functional analysis demonstrates that a worker can operate completely within statutory hours while still exhibiting a fatigue trajectory f (t) that penetrates the unsafe region of the functional space, causing severe cognitive degradation (Dawson et al., 2021; Sprajcer et al., 2022).

A. Tort Liability and Proportional Allocation

When a fatigue-induced error culminates in an incident, courts must determine if the agent breached their duty of care. By evaluating the norm f p, the legal system can quantitatively distinguish between:

1. **Individual Negligence:** Where the agent obtained insufficient sleep despite being afforded ample rest opportunities by their employer, driving up their individual fatigue profile (Jaydarifard et al., 2023; Van Dongen et al., 2003).
2. **Systemic Corporate Liability**: Where the organizational shift scheduling and economic optimization functions force the fatigue trajectory f (t) to inevitably exceed fthresh(t) (Dawson et al., 2011; Jaydarifard et al., 2023; Sprajcer et al., 2022).

B. The Role of Defenses and Mitigation via FRMS

If an organization implements a robust, data-driven Fatigue Risk Management System (FRMS), the system acts as a continuous monitoring constraint that alters the properties of the operator L (Sprajcer et al., 2022). In a legal dispute, proving the existence of active, multi-layered fatigue mitigation protocols lowers the organizational parameter weight in Φ(P), effectively shifting liability away from punitive damages and toward a shared-risk model (Dawson et al., 2021; Gander et al., 2011; Sprajcer et al., 2022). This mathematical framework aligns legal accountability with empirical sleep science, encouraging industries to treat fatigue as a predictable and mathematically manageable hazard.

V. CONCLUSION

This paper presents a novel interdisciplinary framework that integrates mathematical functional analysis with legal reasoning to address the growing challenge of fatigue-related risk in modern workplaces. By representing fatigue trajectories as functions in infinite-dimensional Banach spaces, the proposed model offers a rigorous mechanism for linking empirical fatigue science with concepts of duty of care and liability within tort and regulatory systems. The framework demonstrates that fatigue should not be viewed solely as an individual condition or a matter of compliance with prescribed work-hour limits, but rather as a dynamic systemic phenomenon influenced by organizational decisions, operational environments, and risk-management practices.

The analysis highlights the limitations of traditional legal approaches that rely heavily on static shift-hour regulations and proposes a more evidence-based assessment of responsibility grounded in measurable fatigue risk. By incorporating biomathematical fatigue models into legal evaluation, the framework supports a transition toward systemic accountability, where employers, regulators, and organizations are assessed according to their capacity to anticipate, monitor, and mitigate fatigue-related hazards.

Future research may extend this framework through empirical validation, integration with real-time fatigue monitoring technologies, and the development of computational tools for legal and regulatory decision-making. Ultimately, the proposed approach contributes to a deeper alignment between scientific understanding and legal standards, fostering safer workplaces and more effective mechanisms for managing fatigue-related risks in complex operational settings.

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