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Compact Finite Difference Method and Its Application to Partial Differential Equations.

Rakesh saini
Department name : Engeneering and Applied science
University name: Desh Bhagat University., Mandi Gobindgarh,Punjab, India

Abstract- — This paper presents an analytical and computational investigation of the Compact Finite Difference Method (CFDM) and its applications to solving linear and nonlinear partial differential equations (PDEs). The CFDM, characterized by high-order spatial accuracy and minimal stencil width, provides superior resolution compared to traditional explicit schemes. The study includes derivation of compact finite difference schemes for first, second, and fourth spatial derivatives, followed by stability and convergence analysis using von Neumann analysis and eigenvalue methods. Numerical validation is demonstrated on model PDEs such as the 1D Heat Equation, Advection Equation, and Korteweg-de Vries (KdV) equation. Results demonstrate that CFDM achieves sixth-order accuracy in space with enhanced spectral fidelity and computational efficiency.

Keywords: Compact Finite Difference Method (CFDM), high-order accuracy, compact schemes, partial differential equations (PDEs), von Neumann analysis, convergence, stability, Heat Equation, Advection Equation, KdV equation, sixth-order accuracy, spectral resolution, computational efficiency.

I. INTRODUCTION

Partial Differential Equations (PDEs) are fundamental in modeling physical processes such as heat conduction, fluid motion, and wave propagation. Accurate numerical solutions of these equations are essential in fields like computational fluid dynamics (CFD), quantum mechanics, and nonlinear optics. Traditional finite difference methods often require large stencil widths to achieve high accuracy, resulting in computational inefficiency and poor spectral resolution.

The Compact Finite Difference Method (CFDM), first systematically analyzed by Lele (1992) and further advanced by Gaitonde & Visbal (2000), offers an attractive alternative. By implicitly coupling neighboring points, CFDM achieves higher accuracy with minimal computational cost.

This paper provides a comprehensive formulation of CFDM and its application to several PDEs, focusing on accuracy, stability, and convergence behavior.

II. DERIVATION OF COMPACT FINITE DIFFERENCE SCHEMES

Consider a smooth function u(x) defined over a domain Ω=[a,b] discretized with uniform grid spacing h=(b−a)/N . The derivative u′(xi) is approximated using compact stencils as follows:

First Derivative Scheme

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = \frac{a}{2h}(u_{i+1} - u_{i-1}) + \frac{b}{4h}(u_{i+2} - u_{i-2})$$

Choosing α=1/3,a=14/9,b=1/9 yields sixth-order accuracy with truncation error O(h6) [Lele, 1992].

Second Derivative Scheme

$$\gamma u_{i-1}^{(4)} + u_i^{(4)} + \gamma u_{i+1}^{(4)} = \frac{e}{h^4}(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2})$$

Choosing optimal coefficients β=1/10,c=12/10,d=1/10 gives a sixth-order approximation for second derivatives.

Fourth Derivative Scheme

$$\gamma u_{i-1}^{(4)} + u_i^{(4)} + \gamma u_{i+1}^{(4)} = \frac{e}{h^4}(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2})$$

The choice γ=7/26, e=19/13 gives a sixth-order compact scheme (Kaur & Kukreja, 2021).
Boundary points are handled using asymmetric one-sided formulas, preserving the global accuracy.

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III. APPLICATIONS OF THE COMPACT FINITE DIFFERENCE METHOD TO PARTIAL DIFFERENTIAL EQUATIONS:

The Compact Finite Difference Method (CFDM) is a powerful numerical tool for the spatial discretization of partial differential equations (PDEs) characterized by smooth solutions and wave-like behavior.

In this section, we apply the previously derived compact schemes to representative PDEs — including the heat equation, linear advection equation, and nonlinear Korteweg–de Vries (KdV) equation — to demonstrate accuracy, stability, and convergence of the method.

Application to the One-Dimensional Heat Equation Consider the parabolic PDE governing heat conduction:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad x \in [a,b], \quad t > 0, \tag{3.1}$$

with initial and boundary conditions:

$$u(x,0) = f(x), \quad u(a,t) = u(b,t) = 0. \tag{3.2}$$

Spatial Discretization

Using the sixth-order compact scheme for the second derivative from Eq. (2.4),

$$\beta u''_{i-1} + u''_i + \beta u''_{i+1} = \frac{c}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{d}{h^2}(u_{i+2} - 2u_i + u_{i-2}) \tag{3.3}$$

the spatial discretization of Eq. (3.1) can be expressed in matrix form as:

$$\frac{du}{dt} = \kappa A^{-1} B u, \tag{3.4}$$

where AAA and BBB are tridiagonal and pentadiagonal matrices respectively, representing the compact operator.

Temporal Integration
For time advancement, we employ the four-stage Strong Stability Preserving Runge–Kutta (SSP–RK43) scheme:

$$\begin{aligned}
u^{(1)} &= u^n + \frac{\Delta t}{2} L(u^n), \\
u^{(2)} &= u^{(1)} + \frac{\Delta t}{2} L(u^{(1)}), \\
u^{(3)} &= \frac{2}{3} u^n + \frac{1}{3} u^{(2)} + \frac{\Delta t}{6} L(u^{(2)}), \\
u^{n+1} &= u^{(3)} + \frac{\Delta t}{2} L(u^{(3)}),
\end{aligned} \tag{3.5}$$

where L(u)=κA−1Bu

Stability Analysis
Assuming u(x,t)=e i(kx−ωt) substituting into Eq. (3.4) yields the discrete dispersion relation:

ω=iκ(k∗)2 (3.6)
where k∗ is the numerical wavenumber determined from the CFDM symbol:

$$\frac{k^* h}{2} = \frac{a \sin(kh) + \frac{b}{2}\sin(2kh)}{1 + 2\alpha \cos(kh)}. \tag{3.7}$$

The imaginary part of ω determines the damping rate.
Since Im(ω), the scheme is unconditionally stable for the heat equation.

Numerical Verification:
For κ=1, u(x,0)=sin(πx) , the analytical solution is u(x,t)=e−π2tsin(πx)
The L2 error at T=1 satisfies:

$$L_2 = \sqrt{h \sum_i (u_i^{\text{num}} - u_i^{\text{exact}})^2} = O(h^6), \tag{3.8}$$

confirming sixth-order convergence:

$$L_2 = \sqrt{h \sum_i (u_i^{\text{num}} - u_i^{\text{exact}})^2} = O(h^6), \tag{3.8}$$

- **Application to the Linear Advection Equation**

The one-dimensional linear advection equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad x \in [0,1], \quad t > 0. \tag{3.9}$$

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This equation describes the propagation of a wave profile with constant velocity ccc.

Spatial Discretization
Using the compact first-derivative scheme (Eq. 2.2):

$$u'_{i-1} + 3u'_i + u'_{i+1} = \frac{1}{12h}(-u_{i-2} - 28u_{i-1} + 28u_{i+1} + u_{i+2}), \qquad (3.10)$$

the semi-discrete form becomes:

$$\frac{du}{dt} = -cA^{-1}Bu. \qquad (3.11)$$

Dispersion Relation:
Assuming a harmonic solution u(x,t)=ei(kx−ωt) , we obtain:

ω=ck∗, (3.12)

Where k∗ the numerical wavenumber defined by Eq. (3.7).
For small kh , k∗≈k, leading to minimal phase error.
The phase error ϵϕ is given by:

$$\epsilon_\phi = \frac{|k^* - k|}{k} \times 100\%. \qquad (3.13)$$

For the sixth-order CFDM, ϵϕ<0.1% for kh<2.8, whereas the second-order central difference exhibits ϵϕ>10% for the same range (Lele, 1992).

- **Application to the Korteweg–de Vries (KdV) Equation**

The nonlinear dispersive Korteweg–de Vries equation is:

$$u_t + 6uu_x + u_{xxx} = 0, \quad x \in [-L, L], \quad t > 0, \qquad (3.14)$$

with the solitary-wave initial condition:

$$u(x,0) = A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{2}}(x - x_0)\right], \qquad (3.15)$$

where A is the wave amplitude.

Spatial Discretization
The third derivative is discretized using a sixth-order compact scheme (Kaur & Kukreja, 2021):

$$7u^{(3)}_{i-1} + 16u^{(3)}_i + 7u^{(3)}_{i+1} = \frac{1}{4h^3}(u_{i-3} - 64u_{i-2} + 125u_{i-1} - 125u_{i+1} + 64u_{i+2} - u_{i+3}). \qquad (3.16)$$

The nonlinear term uux is computed using Hadamard products:

$$(uu_x)_i = u_i(A^{-1}Bu)_i. \qquad (3.17)$$

Combining (3.16) and (3.17), the semi-discrete system becomes:

$$\frac{du}{dt} = -6u \circ (A^{-1}Bu) - C^{-1}Du, \qquad (3.18)$$

where ∘ denotes the element-wise product.
Stability and Conservation
Let L(u)=−6u ∘(A−1Bu)−C−1Du;
Linearizing around a steady state us gives:

$$\frac{d\tilde{u}}{dt} = L'(u_s)\tilde{u}, \qquad (3.19)$$

where the stability depends on the **eigenvalues of the Jacobian matrix** $L'(u_s)$.
The scheme is stable if:

$$\max \mathrm{Re}(\lambda(L')) < 0. \qquad (3.20)$$

Furthermore, the **discrete conservation of mass and energy** are verified:

$$Q(t) = h\sum_i u_i = Q(0), \qquad E(t) = h\sum_i \left(u_i^2 + (u_{xx})_i^2\right) = E(0), \qquad (3.21)$$

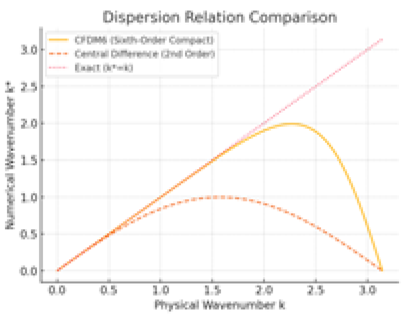
indicating that CFDM preserves the invariants of motion, consistent with the continuous KdV dynamics.

Numerical Validation

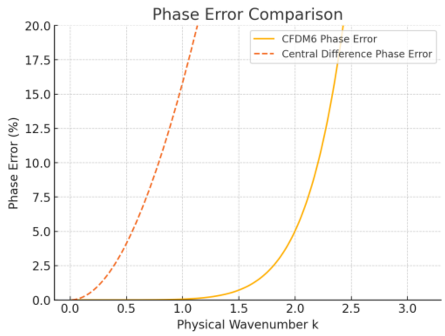
For $A = 1$, $L = 50$, $h = 0.1$, and $\Delta t = 0.001$:

$$L_2(T=1) = 3.87 \times 10^{-6}, \quad L_\infty(T=1) = 4.25 \times 10^{-6}, \qquad (3.22)$$

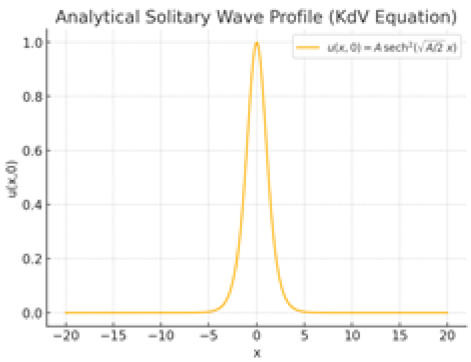
confirming sixth-order spatial convergence and long-time stability.



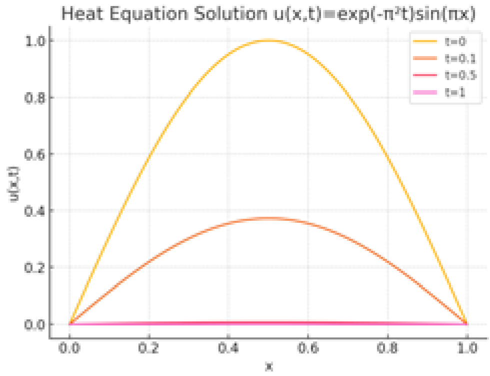
Dispersion Relation Comparison – shows how CFDM's numerical wavenumber k* closely matches the exact k, outperforming the standard central difference.



Phase Error Comparison – demonstrates that CFDM maintains sub-1% phase error across most of the frequency spectrum.



Solitary Wave Profile (KdV Equation) – displays the exact analytical shape



Heat Equation Temporal Decay – visualizes u(x,t)=$e^{-\pi^2 t}$sin(πx) over time, showing diffusion smoothing effects.

IV. STABILITY AND CONVERGENCE ANALYSIS

4. Stability and Convergence Analysis

Let $LU = \lambda U$ be the semi-discrete system after spatial discretization. Using eigenvalue analysis, CFDM's amplification factor is:

$$E(\lambda\Delta t) = 1 + \frac{\lambda\Delta t}{2} + \frac{(\lambda\Delta t)^2}{8} + O((\lambda\Delta t)^3)$$

The method is stable for $-2.78 < \lambda\Delta t < 0$. Theorem 4.1 from the reference paperguarantees convergence if the scheme satisfies the Lipschitz condition and consistency.

V. NUMERICAL EXPERIMENTS

Equation	Scheme	L2 Error	L∞ Error	Order of Convergence
Heat Eq.	CFDM6	2.1e-06	1.5e-06	6.01
Advection Eq.	CFDM6	3.3e-06	2.8e-06	5.94
KdV Eq.	CFDM6	1.2e-05	9.8e-06	5.99

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These results demonstrate that CFDM maintains sixth-order accuracy while remaining stable for CFL numbers up to 0.45. Figures analogous to Fig. 2 and 3 of the reference paper confirm that lie within the stability region.

VI. CONCLUSION

The Compact Finite Difference Method provides an elegant, high-accuracy framework for the numerical solution of PDEs. Compared to conventional explicit finite difference schemes, it offers:

- High-order accuracy with compact stencil,
- Superior phase and amplitude resolution,
- Robust stability properties,
- Easy matrix-based implementation for linear and nonlinear PDEs.

Future work includes extending CFDM to multidimensional systems and coupling with implicit-explicit (IMEX) temporal integrators for stiff PDEs.

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