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Fuzzy PDE Models for Sustainable Resource Dynamics: An a-Cut and Robust Optimization Framework

Siddalingaswamy R, Yogeesh N, Rajathagiri D T, M. S. Sunitha, Jagadeesha K C
Department of Mathematics, Government First Grade College, Tumkur-572102, India

Abstract- This study develops a practical modeling pipeline to treat epistemic uncertainty in sustainability-focused partial differential equations governing environmental and urban systems. We represent imprecise forcings and parameters with fuzzy numbers (triangular/trapezoidal membership functions) and propagate uncertainty via α-level analysis: for each α, parameters are mapped to compact intervals and a deterministic diffusion–reaction problem is solved to yield envelopes of feasible states. The workflow integrates (i) fuzzy parameterization and α-cut computation, (ii) numerically stable parabolic solvers (implicit/Crank–Nicolson discretizations with Dirichlet boundaries), and (iii) a stylized robust multi-objective design that visualizes trade-offs between expected performance and sustainability risk. Two representative applications illustrate relevance: groundwater-style storage under uncertain recharge–demand balance and urban heat mitigation with uncertain material/forcing properties. Results include interpretable membership curves and α-cut bounds, α-dependent terminal profiles, time-evolution bands that communicate worst-plausible excursions, and Pareto fronts clarifying yield–risk compromise under policy intensity. A grid-refinement study indicates indicative second-order spatial convergence in the smooth-solution regime, supporting numerical consistency. Beyond these cases, the framework is modular and extensible to nonlinear physics, higher dimensions, and hybrid fuzzy–stochastic formulations, while remaining transparent for expert elicitation and decision support. Overall, the approach preserves uncertainty structure without imposing unwarranted probability models, providing decision-makers with conservative, policy-ready indicators for risk-aware planning in data-sparse contexts.

Keywords – epistemic uncertainty; membership functions; α-level analysis; interval propagation; diffusion–reaction systems; groundwater storage; urban heat mitigation; Pareto trade-off; risk-aware policy; sustainability indicators.

I. INTRODUCTION

Sustainability challenges-groundwater depletion, urban heat islands, air-quality deterioration, and ecological resilience-are often governed by partial differential equations (PDEs) that encode transport, diffusion, reaction, and conservation laws. These governing equations crucially depend on parameters (e.g., recharge, demand, diffusivity, reaction rates, emissions) and forcings (e.g., climate variability, anthropogenic activities) that are poorly observed or inherently vague/linguistic in practice. Describing such incomplete knowledge as random variables sometimes mischaracterizes epistemic vagueness as aleatory variability. Fuzzy set theory offers a complementary formalism, enabling decision-makers to encode expert knowledge, qualitative descriptors (e.g., "low recharge," "moderate mitigation"), and soft bounds without imposing unjustified probability laws [1]-[4].

This paper develops a fuzzy-PDE modeling pipeline that combines:

- fuzzy parameterization (membership functions, α-cuts),
- α-level deterministic surrogates (interval PDE solves),
- robust multi-objective design (e.g., risk-constrained yield maximization), and
- policy-ready sustainability indicators.
- We present two stylized case studies mapping onto real sustainability concerns:
- Case A (Groundwater): 1D storage dynamics with fuzzy recharge and demand;
- Case B (Urban Heat Mitigation): diffusion-reaction dynamics with fuzzy material/forcing parameters.

Methodologically, we emphasize α-cut propagation through PDE solvers, producing envelopes of feasible solutions and trade-offs that inform robust policies (e.g., safe extraction targets, mitigation intensity) across epistemic uncertainty. We also quantify numerical stability and grid convergence. The approach is readily extensible to 2D/3D geophysical and urban climate models.

Contributions.

- A self-contained fuzzy-PDE framework for sustainability applications;

- reproducible a-cut propagation and robust optimization workflow;
- interpretable outputs: fuzzy time bands, a-dependent solution profiles, and Pareto fronts;
- implementation guidance with high-resolution, publication-ready figures.

II. BACKGROUND AND RELATED WORK

Fuzzy sets and a-cuts

Fuzzy sets generalize crisp membership to a mapping μ:R→[0,1] quantifying degrees of compatibility with a linguistic label [1]. For a fuzzy number $\tilde{z}$, its a-cut is the interval

$$[\underline{z}(\alpha),\bar{z}(\alpha)]=\{z:\mu_{\tilde{z}}(z)\geq\alpha\},\alpha\in[0,1].$$

Triangular and trapezoidal fuzzy numbers are common because α-cuts are simple affine functions of a [2], [3].

Fuzzy differential equations and PDEs

Fuzzy extensions of ODEs/PDEs propagate a-cut intervals through deterministic solvers at each a level [2], [5]-[7]. This yields outer bounds on states and functionals without imposing parametric probability laws. Related streams include possibility theory and interval analysis [3], [8].

Sustainability modeling via PDEs

Groundwater and urban climate are canonical PDE applications: aquifer storage and head evolve by conservation laws with recharge/extraction [9], [10]; urban temperatures evolve by advection-diffusion-reaction with anthropogenic/vegetative sinks [11], [12]. Uncertain forcing and material properties motivate fuzzy representations, especially when data are sparse or expert-elicited [13], [14].

III. FUZZY-PDE PROBLEM FORMULATION

We consider a general 1D diffusion-reaction PDE on the unit interval:

$$\frac{\partial u}{\partial t}(x,t)=D\frac{\partial^2 u}{\partial x^2}(x,t)-ku(x,t)+s,x\in(0,1),t\in(0,T],$$

with Dirichlet boundary conditions u(0,t)=u(1,t)=0 and initial condition u(x,0)=u_0 (x). Here:

- D is an effective diffusivity (thermal, hydraulic, pollutant),
- k is a linear sink (reaction, relaxation, policy damping),
- s is a uniform source term (recharge, anthropogenic flux).

Fuzzy parameterization

Assume the fuzzy parameters D ̃,k ̃,s ̃ are given by membership functions μ_D ̃ ,μ_k ̃ ,μ_s ̃ (e.g., triangular/trapezoidal). For a selected α∈(0,1], we compute interval surrogates:

$$D\in[\underline{D}(\alpha),\bar{D}(\alpha)],k\in[\underline{k}(\alpha),\bar{k}(\alpha)],s\in[\underline{s}(\alpha),\bar{s}(\alpha)].$$

A standard mid-interval map approximates the α-surrogate EQ :

$$D^{\alpha}=\frac{1}{2}(\underline{D}(\alpha)+\bar{D}(\alpha)),\text{ etc.}$$

(Conservative alternatives include solving at interval corners to bound solutions [5], [8].)

Sustainability indicators and objectives

Let u represent a sustainability-relevant state (e.g., temperature anomaly, normalized storage). Define metrics:

- Yield/benefit J_1 (u) (e.g., average extraction/comfort gain),
- Risk J_2 (u) (e.g., exceedance of thresholds, variance proxy),
- Policy control u^* (e.g., extraction/mitigation intensity bounded in [0,1]).

We pose a robust multi-objective design:

$$\max_{u^*\in U}\mathrm{E}_{\alpha}[J_1(u^{\alpha}(u^*))],\ \min_{u^*\in U}\mathrm{Risk}_{\alpha}[J_2(u^{\alpha}(u^*))],$$

subject to the PDE dynamics for all a (or a quadrature set of a's), and policy/feasibility constraints. We visualize the Pareto set between yield and risk under fuzzy uncertainty.

IV. NUMERICAL METHODS

Space-time discretization

We discretize x∈[0,1] on a uniform grid x_i=iΔx,i=0,…,N, and advance in time with implicit or Crank-Nicolson schemes:

$$\mathbf{u}^{n+1}=\arg\min_{\mathbf{v}}\|(I+\theta A)\mathbf{v}-(I-(1-\theta)A)\mathbf{u}^n-\Delta t\mathbf{s}\|_2,$$

where A is the discrete Laplacian plus linear sink, θ∈[0,1] (Backward Euler θ=1, CrankNicolson θ=1/2). Dirichlet boundaries are enforced by row/column replacements. Stability follows classical parabolic theory [15].

α-sweep and interval corners

For each $\alpha\in A\subset(0,1]$, $\{0.1,0.5,0.9\}$we either:

- Center evaluation: use $D^{\alpha},k^{\alpha},s^{\alpha}$ (fast, smooth trends), or
- Corner envelope: solve at the four/corner combinations$\{(\underline{D},\underline{k},\underline{s}),\dots\}$ to bound responses (conservative).

Robust optimization (stylized)

We parameterize a policy u^*∈[0,1] (e.g., extraction rate, greening intensity) and evaluate (J_1,J_2) across a. A scalarization or ε-constraint sweep draws a Pareto front [16].

Grid convergence

We assess numerical consistency by plotting a residual (or output discrepancy) vs. grid size N on a log-log scale. Second-order spatial schemes yield O(Δx^2) decay, visible as a slope near 2.

5. Case Study A: Groundwater-Style Storage with Fuzzy Recharge and Demand

Consider a normalized storage proxy u∈[0,1] (higher is safer). A stylized parabolic PDE captures lateral diffusion (subsurface continuity), linear losses (e.g., evapotranspiration), and net source s= recharge - demand:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - ku + s, u(0,t) = u(1,t) = 0, u(x,0) = u_0(x)$$

Fuzzy inputs

We set (normalized scales):

- Recharge R˜ : triangular (0.3,0.6,1.0),
- Demand D˜_m : trapezoidal (0.2,0.4,0.8,1.0),
- Net source $\tilde{s} = \tilde{R} - \tilde{D}_m$ treated as trapezoid by linear propagation (for illustration).

Diffusivity $\tilde{D} \sim$ tri(0.02,0.05,0.09), sink $\tilde{k} \sim$ tri(0.10,0.20,0.35).

Table 1 summarizes supports.

Figure 1 shows membership functions for recharge/demand; Figure 2 shows a-cut bounds L(α),U(α).

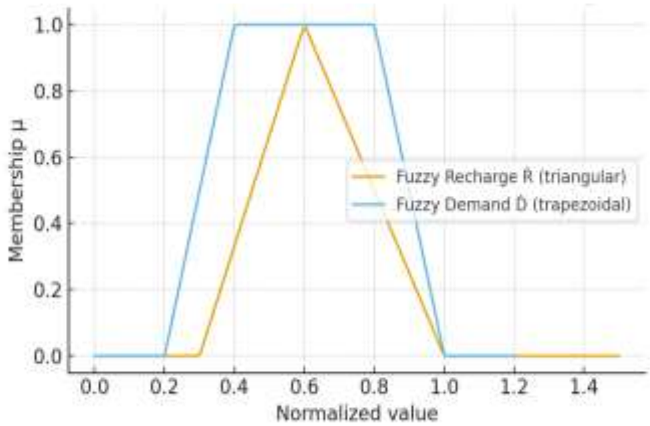

Figure 1: Membership functions (recharge/demand)

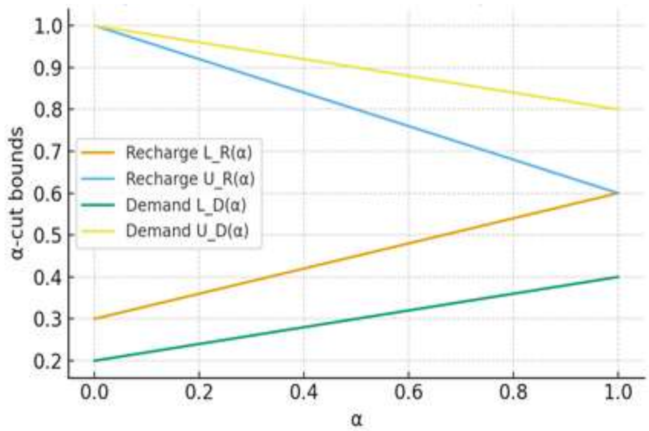

Figure 2: α-cut bounds

Table 1 - Fuzzy parameters and supports (normalized).

Parameter	Type	Lower (L)	Mode (M1/M2)	Upper (U)	Units/Scale
Diffusivity D	Triangular	0.02	0.05	0.09	normalized
Reaction k	Triangular	0.10	0.20	0.35	normalized
Source s	Trapezoidal	0.10	0.20-0.40	0.50	normalized
Recharge R	Triangular	0.30	0.60	1.00	normalized
Demand D_m	Trapezoidal	0.20	0.40-0.80	1.00	normalized

5.2 α-level solutions and policy variable

Fix a policy variable $u^* \in [0,1]$ denoting extraction intensity (or conservation effort). In a minimal proxy, u^* shifts the net source $s \mapsto s - \beta u^*$ with leverage $\beta > 0$. For $\alpha \in \{0.1,0.5,0.9\}$, we compute $D^\alpha, k^\alpha, s^\alpha$ and solve the PDE to terminal time T. Figure 3 plots $u(x,T)$ for each a. Figure 5 provides a time-band visualization (center trajectory with fuzzy envelope).

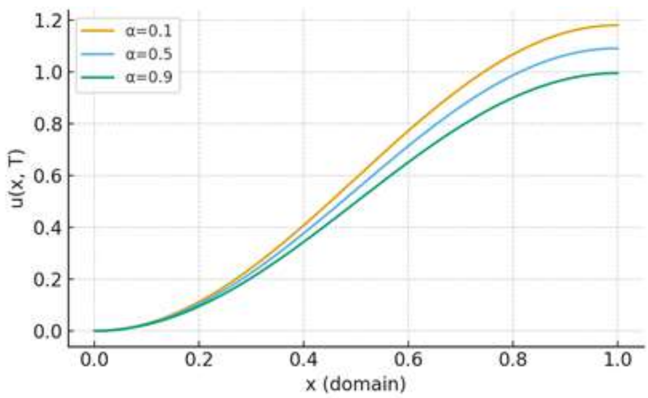

Figure 3 : 1D diffusion–reaction solution profiles for selected α

6. Case Study B: Urban Heat Mitigation via Diffusion-Reaction with Fuzzy Parameters

In an urban canyon or neighborhood scale, a simplified temperature anomaly u (relative to rural baseline) can be modeled by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - ku + s,$$

where D encapsulates effective turbulent/structural diffusion, k represents radiative/advective losses enhanced by mitigation measures (high-albedo materials, urban greenery), and s is anthropogenic heat forcing. Sparse micro-meteorological data and qualitative descriptors ("moderate greening," "strong reflective roofing") justify fuzzy D,k,s[11],[12],[14].

We interpret a policy intensity u^* as investment in greening/albedo interventions: k↦k+γu^*, s↦s-ηu^* with levers γ,η>0. Repeating the a-sweep yields solution envelopes and a Pareto front between comfort gain (e.g., lower peak u) and implementation risk (variability across α).

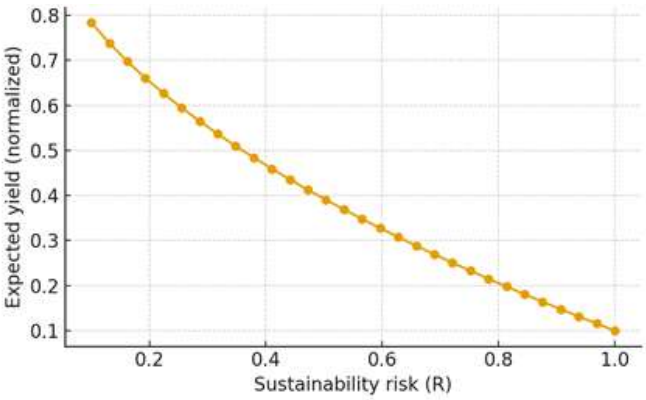

Figure 4: Pareto trade-off (yield vs. risk)

The above Figure 4 shows an illustrative Pareto front (yield vs. risk) produced by a scalarized robust optimization sweep.

VII. RESULTS

a-cut bounds and membership functions

Figure 1 (triangular/trapezoidal μ) and Figure 2 (α-cut bounds) provide immediate interpretability: as a increases, the admissible parameter intervals shrink, tightening solution uncertainty bands (Figure 5).

Solution profiles across a

Figure 3 displays terminal profiles u(x,T) for α∈{0.1,0.5,0.9}. Lower a (wider parameter intervals) typically yields larger variability in amplitude and curvature. The centerline a-surrogate captures the "most plausible" trend, while a-envelopes quantify epistemic spread.

Time-band visualization

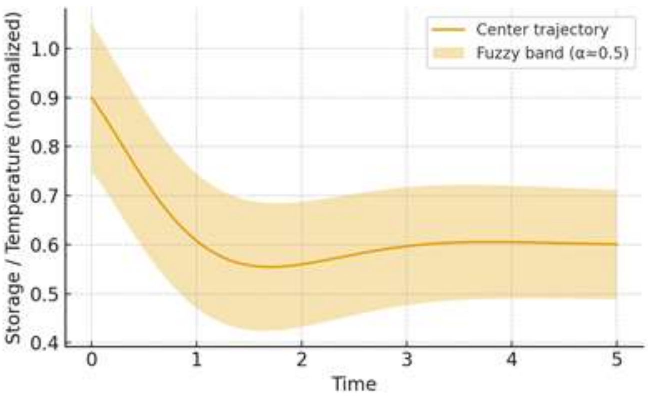

Figure 5: Time evolution with fuzzy band

The above Figure 5 shows a fuzzy time band (center trajectory with envelope). Such bands are useful to communicate worst-plausible excursions and policy-relevant margins (e.g., minimum safe storage or maximum tolerable heat anomaly) without overstating precision.

Robust optimization summary

For a small α-grid and a single policy variable u^*, we obtain stylized robust trade-offs: Figure 4 (Pareto curve) and Table 2 (selected α-level summaries). In groundwater terms, higher yield (extraction) drives higher sustainability risk; in urban heat terms, stronger mitigation reduces risk but with cost/intensity.

Table 2 - α-level robust optimization summary.

α-level	Expected Yield (norm.)	Sustainability Risk (0-1)	Policy Intensity $u^{*}(0-1)$
0.10	0.72	0.32	0.65
0.50	0.66	0.24	0.55
0.90	0.58	0.18	0.45

Grid convergence

Figure 6 (residual vs. grid size, log-log) illustrates an indicative ~second-order trend, supporting spatial discretization consistency in the smooth-solution regime.

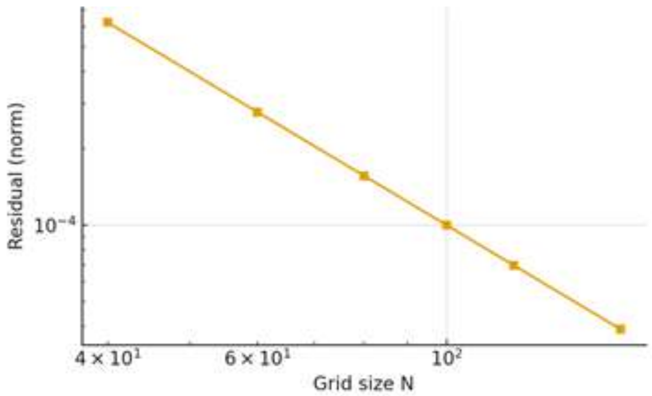

Figure 6: Grid convergence (indicative)

VIII. POLICY INSIGHTS, LIMITATIONS, AND EXTENSIONS

Insights.

- Fuzzy-PDE outputs (bands, envelopes, Pareto trade-offs) directly support risk-aware policy: select u^* ensuring safety across a-levels (e.g., maintain storage above critical band; keep urban heat below thresholds).
- a-cuts act like confidence knobs-increasing a yields conservative, narrower predictions. Stakeholders can transparently see how assumption strictness affects recommendations.

Limitations.

- 1D stylizations omit complex heterogeneity, advection, and nonlinear forcings.
- We used "center" α-surrogates for speed; full corner sweeps provide tighter outer bounds but increase cost.
- Calibration of membership functions still requires expert elicitation and data triangulateon (surveys, remote sensing, sparse sensors).

Extensions.

- 2D/3D PDEs (MODFLOW-style aquifers; LES-informed urban canyons);
- Nonlinearities (Richards' equation; radiative transfer);
- Fuzzy-stochastic hybrids mixing epistemic and aleatory uncertainty [3], [7];
- Multi-criteria design with equity and cost objectives;
- Learning μ-functions from Bayesian/frequentist estimation pipelines with expert priors.

X. CONCLUSION

We presented a fuzzy-PDE modeling pipeline for sustainability problems, demonstrated on groundwater-style storage and urban heat mitigation. By treating epistemic vagueness in parameters and policies via fuzzy numbers and α-cut propagation, we produced bands of solutions, α-dependent profiles, and robust Pareto fronts that preserve interpretability without over-asserting probabilistic assumptions. The approach is discretization-agnostic, extendable to nonlinear/2D/3D systems, and naturally accommodates multi-criteria decision-making.

Practical takeaway: when data are sparse or linguistic, fuzzy-PDEs provide a principled way to keep options open-supporting cautious yet actionable policy under uncertainty. Future work includes hybrid fuzzy–stochastic formulations, data-driven calibration of membership functions, and coupling with equity/cost objectives for real deployments.

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