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$\mathcal{T}$- Purity and $\mathcal{T}_C$ - Purity in Modules

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Abstract- An exact sequence $\mathrm{E}: 0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{B} \longrightarrow \mathrm{C} \longrightarrow 0 \ldots .(1)$ is called $\mathcal{T}-$pure if any torsion $\mathrm{R}-$ module is projective and relative to it and $\mathfrak{F}-$ copure if any torsion free $\mathrm{R}-$ module is injective relative to it. . Since $\mathcal{T}$is closed under factors and $\mathfrak{F}$ is closed under sub-modules. Here Walker's [19] criterion of Co-purity is also applicable in this situation. We also know that $\mathrm{Pext}_{\mathcal{T}}(\mathrm{M},\mathrm{A}) = 0$ if and only if an $\mathrm{R}-$ module M is $\mathcal{T}-$pure projective and $\mathrm{Pext}_{\mathfrak{F}}(\mathrm{A},\mathrm{M}) = 0$ if it is $\mathfrak{F}$ - copure injective for all $\mathrm{A} \subseteq \mathrm{M}$. In particular $\mathrm{Pext}_{\mathcal{T}}(\mathrm{T},\mathrm{A}) = 0$ for all $\mathrm{T} \in \mathcal{T}$. We write the torsion sub-module of $\mathrm{A} \subseteq \mathrm{M}$ by $\sigma(\mathrm{A})$. Walker proved that the class of $\mathrm{I}-$ pure ($\mathcal{J}-$ copure) sequences form a proper class whenever $\mathrm{I}(\mathcal{J})$ is closed under homomorphic images (sub-modules) of an $\mathrm{R}-$ module M and if $\mathrm{I}(\mathcal{J})$ is closed under factors (sub-modules) then for any $\mathrm{I}-$ pure ($\mathcal{J}-$ copure) sequence $\mathrm{E}: 0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{B} \longrightarrow \mathrm{C} \longrightarrow 0$ if $\mathrm{E} \in \pi^{-1}(\mathscr{I})$ ($\mathrm{E} \in \mathrm{i}^{-1}(\mathscr{I})$) and hence in this case the earlier notion of purity coincides with Walker's $\mathrm{I}-$ purity ($\mathcal{J}-$ copurity) . A sequence $\mathrm{E}: 0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{B} \longrightarrow \mathrm{C} \longrightarrow 0$ is $\mathrm{I}-$ pure ($\mathcal{J}-$ copure) if and only if given $\mathrm{C}' \leq \mathrm{C} \in \mathscr{I}$, then there exists$\mathrm{B}' \leq \mathrm{B}$ such that $\mathrm{B}' \cong \mathrm{C}'$ and $\mathrm{A} \cap \mathrm{B}' = 0$. We consider an another stronger notion of purity than the Cohn's purity[11]. If $\mathrm{F}\mathcal{G}$ denotes the class of all finitely generated $\mathrm{R}-$modules, which is closed under factors. We shall try to develope some characterizations of $\mathrm{F}\mathcal{G}-$purity and to determine its relationship with the $\mathcal{T}-$ purity and $\mathcal{T}_C-$ purity in cyclic torsion modules We also derive some relations of absolutely $\vartheta-$ pure modules with it . We try to relate it the with conditions for $\mathcal{T}-$ pure projectivity Teply and Golan [18].. We relativize the above concept and also relate it with finite projectivity of Azumaya [8] with respect to a torsion theory and to study the inter-relationship between these concepts. Finite $\sigma-$projectivity, $(\mathrm{FG},\sigma)-$ pure flatness, cyclically $\sigma-$ pure projectivity and cyclically $\sigma-$ pure flatness, the concept of locally $\sigma-$ projectivity and locally $\sigma-$ splitness are also considered here and we study its inter-relationship with $(\mathrm{FG},\sigma)-$ purity and semi-simple module.

Index Terms- $\mathrm{R}-$ modules, $(\mathrm{FG},\sigma)-$ purity, $\sigma-$ pure projective/injective $\mathrm{R}-$modules, $\mathscr{I}-$ pure ($\mathcal{J}-$ copure, $\mathcal{F}\mathcal{G}-$flat modules, cyclically $\sigma-$ pure projectivity, $\sigma-$ pure injectivity. Subject classification: 16D99

I. INTRODUCTION

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In the given sense we say that if any short exact sequence with M as the first (respectively second, third) position is pure with respect to the purity if an $R-$ module M is absolutely pure, (respectively regular, flat). Now we take a right $R-$ module N and $\bigoplus_J R \overset{\mu}{\to} \bigoplus_I R \longrightarrow N \longrightarrow 0$. We take all the column finite sub-matrices associated with $\mu = (r_{ij})$ which is an $i \times j$ matrix determined by M. The class of all co-kernels of the right $R-$ maps $\mu = (r_{ij})$ between $\bigoplus_J R$ and $\bigoplus_I R$ which is induced by these sub-matrices is denoted by $\wp(N)$. Now we consider all row finite sub-matrices of the matrix and take co-kernels of all left $R-$maps between $\bigoplus_I R$ and $\bigoplus_J R$ induced by these sub-matrices and this class of left $R-$ modules is denoted by $\mathcal{L}(N)$. An exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{T}-$pure ($\mathcal{J}-$ copure) if any torsion (torsion free) module is projective (injective) relative to it. Since $\mathcal{T}(\mathcal{J})$ is closed under factors (sub-modules). In this situation Walker's criterion of Co-purity is applicable. The notation of an $R-$ module M is $\mathcal{T}-$pure projective ($\mathcal{J}-$ copure injective) if and only if $Pext_{\mathcal{T}}(M,A) = 0$

$(Pext_{\mathfrak{F}}(A, M) = 0)$ for all $A \subseteq M$. In particular $Pext_{\mathcal{T}}(T, A) = 0$ for all $T \in \mathcal{T}$.

We denote the torsion sub-module of $A \subseteq M$ by $\sigma(A)$. Walker proved that the class of $\mathcal{T} -$ pure ($\mathcal{J} -$ copure) sequences form a proper class whenever $\mathcal{T}(\mathcal{J})$ is closed under homomorphic images (sub-modules) of an $R -$ module M and if $\mathcal{T}(\mathcal{J})$ is closed under factors (sub-modules) then for any $\mathcal{T} -$ pure ($\mathcal{J} -$ copure) sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and $E \in \pi^{-1}(\mathcal{I})$ $(E \in i^{-1}(\mathcal{J}))$. Hence, in this case Walker's $\mathcal{T} -$ purity ($\mathcal{J} -$ copurity) coincides with the earlier notion of purity. We also study about class of $R -$modules dual to the modules of B. A sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T} -$ pure ($\mathcal{J} -$ copure) if and only if given $C' \leq C \in \mathcal{I}$, there exists $B' \leq B$ such that $B' \cong C'$ and $A \cap B' = 0$. We consider an another stronger notion of purity than the **Cohn**'s **purity**[11]. If $F\mathcal{G}$ denotes the class of all finitely generated $R -$modules, which is closed under factors. We shall try to develop some characterizations of $F\mathcal{G} -$purity and to determine its relationship with the $\mathcal{T} -$ purity and $\mathcal{T}_C -$ purity in cyclic torsion modules We also derive some relations of absolutely $\vartheta -$ pure modules with it . We try to relate it the with conditions for $\mathcal{T} -$ pure projectivity **Teply and Golan [18]**.

We relativize the above concept and also relate it with finite projectivity of **Azumaya** [8] with respect to a torsion theory and to study the inter-relationship between these concepts. Finite $\sigma -$projectivity, $(FG, \sigma) -$ pure flatness, cyclically $\sigma -$ pure projectivity and cyclically $\sigma -$ pure flatness, the concept of locally $\sigma -$ projectivity and locally $\sigma -$ splitness are also considered here and we study its inter-relationship with $(FG, \sigma) -$ purity and semi-simple module.

An $R -$ module M is called locally projective if given a map $g: M \longrightarrow B$ and a finitely generated sub-module F of M, if there exists a map $g': M \longrightarrow A$ such that($\pi o g')|F = g|F$, that is

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & M \\
A \arrow[r, "\vec{\pi}"] & B \arrow[r] & 0
\end{tikzcd}$$

We know that all locally projective modules are flat and this class lies strictly between and projective modules and flat modules. We relate these concepts to $\mathcal{F}\mathcal{G} -$ purity which is same as finite splitness with respect to a hereditary torsion theory which is given by a connection of $(\mathcal{F}\mathcal{G}, \sigma) -$ purity with $\mathcal{T}_{\mathcal{F}} -$ purity (torsion purity) of a left exact torsion radical σ. We also relativize the concept of finite (cyclic) $\sigma -$ extension and finitely (cyclically) $\sigma -$splitness. In this present paper we relativize the concept of $\mathcal{F}\mathcal{G}$ - purity and $\sigma -$ injectivity and $\sigma -$projectivity of $R -$ modules. We observe that the torsion $\sigma -$ purity of **Bhattacharya** and **Choudhury**[9] reduces to usual purity, $(\mathcal{F}\mathcal{G}, \sigma) -$ splitness and cyclic $\sigma -$ purity becomes purity relative to cyclic modules that is singly (cyclically) $\sigma -$ pure. We also give the results of the characterization of a Noetherian like condition on the torsion theory. Here $\mathcal{T}_1$- purity coincides with the usual purity (**Cohn purity**). In this paper we try to develop the theory of $\sigma -$purity relative to a torsion theory $(\mathcal{T}, \mathcal{T}_1)$ radical σ which is weaker than $\mathcal{T} -$purity but it gives the generalization of usual purity (Cohn purity) and also gives a $\sigma -$generalization of regular modules.

Definition:

1. A left $R -$ module M is called **finitely co-generated** if and only if for every set $\{U_i \mid i \in I\}$ of submodules $U_i \subseteq M$ with $\cap_{i \in I} U_i = 0$, there exists a finite subset $\{U_i \mid i \in I_0\}$ that is $I_0 \subset I$ and I_0 is finite with $\cap_{i \in I} U_i = 0$. In another way we can say A module M is said to be **finitely co-generated** if it is co-generated by the family $\{E(S_{i \in I}) \mid S_{i \in I}$are simple modules $\}$ finitely. That is $E(M) = \oplus_{i=1}^{n} E(S_i)$ where $S_{i \in I}$ simple modules are, it is not necessary that they are non- isomorphic.

2. An $R -$ module M is called **co-cyclic module** if it is contained in $E(S)$ for some simple module S, where $E(S)$ is a family of co-generators for each R module M.

3. In the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
M \arrow[r] & N
\end{tikzcd}$$

Where $\mu: A \longrightarrow M$ $f: A \longrightarrow B$; $\varphi: M \longrightarrow N$, and $g: B \longrightarrow N$ are maps. The pair (φ, g) is called **push out** of the pair (μ, f) if and only if for every pair (φ', g') with $\varphi': M \longrightarrow$X, $g': B \longrightarrow X$ and $(\varphi' o\, \mu) = (g' o f)$, there exists a unique map $\sigma: N \longrightarrow X$ such that $(\sigma o g) = g'$.

4. The pair (ϕ, f) is called **pullback** of the pair (ψ, g) if and only if for every pair (ϕ', f') with $\phi': Y \rightarrow M$, $f': Y \rightarrow B$ and $(\psi o \phi') = (g o f')$, there exists a unique map $\tau: Y \rightarrow A$ such that $(f o \tau) = f'$ and $(\phi o \tau) = \phi'$.

5. If there is an exact sequence $M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\pi} M \longrightarrow 0$ where M_0 and M_1 are free $R -$modules with finite bases then an $R -$ module M is called **finitely presented** .

6. Suppose R is a ring and if for every exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ and the transformed sequence $0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$ is exact.
then a left $R -$ module M is called **flat module**.

7. A ring R iscalled **hereditary** if and only if every ideal of R is a projective module.

8. If M be an R −module , the sum of all simple sub-modules of M is said to be the **socle of M** and it is denoted by $s(M) = \{x \in M|Ann(x)$is a finite intersection of maximal right ideals $of\ R\}$. That is if $x \in s(M)$, then xA is a direct sum of a finite number of simple modules where A is a semi-simple ring.

9.An R module S has only submodules $\{0\}$and S then it is called simple. A module M is a sum of simple sub-modules of M then it is called **semi-simple**.

10. A **torsion theory** is a pair $(\mathfrak{I}, \mathfrak{F})$ of classes of modules satisfying:

- (i). $Hom(T, F) = 0, \forall\, T \in \mathfrak{I}$ and $F \in \mathfrak{F}$
- (ii). $Hom(L, F) = 0, \forall\, F \in \mathfrak{F} \Rightarrow L \in \mathfrak{I}$
- (iii). $Hom(T, N) = 0, \forall\, T \in \mathfrak{I} \Rightarrow N \in \mathfrak{F}$

The classes $\mathfrak{I}$ *and* $\mathfrak{F}$ are known as **torsion** and **torsion free** classes associated with a torsion theory$(\mathfrak{I}, \mathfrak{F})$. A torsion theory $(\mathfrak{I}, \mathfrak{F})$ is called **hereditary** if and only if $\mathfrak{I}$ is closed under homomorphic images, direct sums, extensions and sub-modules. Similarly, $\mathfrak{F}$ is closed under submodules, direct products, extensions and injective envelopes.

11. A given $\sigma -$ pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, there exists a map $h: P \longrightarrow B$ such that $poh = f$ where as $p: B \longrightarrow C$ be an onto homomorphism then a left $R - module$ P is called **$\sigma -$ pure projective** module if it is projective relative to every $\sigma -$ pure epimorphism..

12. A left $R - module$ Q is called **finitely $\sigma -$ pure injective** if it is $(\mathcal{FG}, \sigma)$- pure in every pure extension of Q, that is if $0 \longrightarrow Q \longrightarrow Q' \longrightarrow Q'' \longrightarrow 0$ is a pure exact sequence then it is $(\mathcal{FG}, \sigma)$- pure also.Similarly, Q is called **cyclically $\sigma -$ pure injective** if it is cyclically $\sigma -$ pure in every pure extension of it.

13. A sub-module A of an R-module B is said to be **closed** if $B|A$ is torsion free and it is said to be dense if $B|A$ is torsion. Any closed submodule A of an R-module B is $\mathcal{T}$ −pure.

14. A sub-module $A \subseteq M$ is said to be **$\mathfrak{I} -$ essential** if it intersects every torsion sub modules of M.

15. A sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called **$\mathcal{T} -$ pure** if A is a direct summand of D whenever $A \subseteq D \subseteq B$ and $|A \in \mathcal{T}$ *and* where $\mathcal{T}$is a class of modules .

Walker proved that the class of $\mathcal{T} -$ pure sequences form a proper class whenever $\mathcal{T}$ is closed under homomorphism of an $R -$ module M and if $\mathcal{T}$ is closed under factors then for any $\mathcal{T} -$ pure sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E \in \pi^{-1}(\mathcal{T})$ and hence in this case Walker's $\mathcal{T} -$ purity coincides with the earlier notion.

Proposition1.1: If $\mathcal{T}$ is closed under factors then a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T} -$ pure if and only if given $C' \leq C \in \mathcal{T}$ there exists $B' \leq B$ such that $B' \cong C'$ and $A \cap B' = 0$.

Definition1.2: We say that a submodule A is (μ, σ) −pure in an $R -$ module B, if any system of linear equation $\sum r_{ij}\, x_j = a_i$ given by the row finite matrix μ in A, whenever it is solvable in B in the form $x_j = b_j$ for which there are left ideals $D_i \in D$ where D is the **Gabriel filter** of dense left ideals corresponding to the left exact torsion radical σ, such that $D_j b_j \in A$. The system is also solvable in A that is there are $a'_j \in A$, with $\sum r_{ij}\, a'_j = a_i$ for each $i \in I$ and $j \in J$. This exactly means that given vectors $(b_j) \in \prod_J B$ and $(a_i) \in \prod_I A$ and $\mu(b_j) = a_i$ with $D_j b_j \subseteq A$ for some $D_i \in D$, there exists $(a_j{'}) \in \prod_I A$ such that $\mu(a_j{'}) = a_i$ where the vector $\mu(a_j{'})$ is obtained by matrix product of the row finite matrix μ and column vector $(a_j{'})$. We may rephrase the above condition that a submodule A is (μ, σ) −pure in an $R -$ module B or that B is a (μ, σ) −pure extension of A as follows.

We view the mapping $\mu: \prod_J B \rightarrow \prod_I B$ by left matrix multiplication. Then we have:

Theorem1.3: A submodule A is (μ, σ) −pure in B if and only if $\mu[\prod_j B] \cap \prod_I A \subseteq \mu[\prod_I A]$ whenever B_j' are submodules of B containing A such that A is dense in B_j.

Proof: Any element of the left hand side is of the form $(a_i)_I = \mu((b_j)_J) = \sum r_{ij}\, b_j$ and A dense in B_j means $B_j|A$ is torsion and hence for each element $(b_j + A) \in B_j|A$, there exists $D_j \in D$ such that $D_j(b_j + A) = 0$ that is $D_j(b_j) \subseteq A$.

The following result links (μ, σ) −purity with (M, σ) −purity.
Proposition1.4: Let $\mu = (R_{ij})$ be a row finite $I \times J$ matrix where I and J are arbitrary sets. Then a submodule A is (μ, σ) −pure in a module B if and only if the sequence

$0 \longrightarrow A \longrightarrow B \longrightarrow B|A \longrightarrow 0$ is (M, σ) −pure where $\oplus_i R \overset{\mu'}{\rightarrow} \oplus_j R \longrightarrow M \longrightarrow 0$ is exact with μ' given by the matrix μ.
Definition1.5: A submodule A is $\mathcal{T} -$ **pure** in an $R -$ module M if and only if given a torsion submodule C of $M|A$, there exists a submodule B of M such that $B \cong C$ and $A \cap B = 0$.

$(\mathcal{T}, \mathcal{T}_1)$ denotes a hereditary torsion theory with the corresponding idempotent kernel functor σ.

Definition1.6: A submodule A of an $R-$ module M is called $\mu-$ pure in M where $\mu = (x_{ij})$ if whenever the system of linear equations $\sum r_{ij}\, x_j = a_i,\ i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $D_j \in D$, the associated Gabriel filter for left dense ideals, is solvable in M, it is solvable in A.

Definition1.7: A submodule A of an $R-$ module M is called $\sigma-$ pure in M if whenever a finite system of linear equations in a finite number of variables $\sum r_{ij}\, x_j = a_i,\ i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $D_j \in D$, the associated Gabriel filter for left dense ideals, is solvable in M, it is solvable in A.

Proposition1.8: A submodule A of an $R-$ module$M$ is $\sigma-$ pure in M if and only if A is **(Cohn)**$-$ pure **[11** in the closure of A in M.

Proposition1.9: If $\oplus_i R \overset{\mu'}{\rightarrow} \oplus_j R \longrightarrow M \longrightarrow 0$ is exact then a submodule A of an $R-$ module is $\mu-$pure in M if and only if A is $M-$ pure in the closure of A in M.

Proof: The closure $\bar{A}$ of A is defined by $\bar{A}|A = \sigma(M|A)$. If A is $\mu-$pure in $\bar{A}$, then by **Azumaya**[**8**], A is $\mu-$pure in $\bar{A}$. Then the given a finite system of linear equations in a finite number of variables $\sum r_{ij}\, b_j = a_i$; $i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $D_j \in D$, $(m_j + A) \in \sigma(M|A) = \bar{A}|A$. Hence, $m_j \in \bar{A}$. As A is pure in $\bar{A}$ then there exists $a_j{}' \in A$ such that $\sum r_{ij}\, a'_j = a_i$, and the system is solvable in A.

Conversely, if it is given that a finite system of linear equations in a finite number of variables $\sum r_{ij}\, m_j = a_i$, $i \in I$ with $a_i \in A$ and $m_j \in \bar{A}$ then, $(m_j + A) \in \bar{A}|A = \sigma(M|A)$ there is $D_j \in D$, $D_j(m_j + A) = 0$ that is $D_j(b_j) \subseteq A$ and hence the system is solvable in A and so, A is pure in $\bar{A}$. Hence A is $\mu-$pure in $\bar{A}$ by **Azumaya**[**8**] proposition (1).

Definition1.10: **(1).** An $R-$ module C is said to be $\sigma-$ flat if a submodule A is $\sigma-$ pure in an $R-$ module B whenever $C \cong B|A$.

(2). A submodule A of an $R-$ module B is said to be $(\mu,\sigma)-$ pure if and only if $A \subseteq \bar{A}$ is $\mu-$ pure.

(3). A submodule A of an R- module B *is* $\mathcal{T}$ $-$essential if it intersects every torsion submodule of B.

Proposition1.11: Every torsion free module is $\sigma-$ flat and every torsion $\sigma-$ flat module is flat. Also, every flat module is $\sigma-$ flat of course.

Proposition1.12: A submodule A is closed in B if and only if A is $\mathcal{T}$ $-$pure and $\mathcal{T}$ $-$essential in B.

Proof: If A is closed in B then A is $\mathcal{T}$ $-$pure. Suppose that $A \cap B_1 = \{0\}$ for some $B_1 \subseteq B$ and $B_1 \in \mathcal{T}$. But $B_1 \subseteq \sigma(B)$ and $\sigma(B) = \cap\, C$, where $C \subseteq B$ and $\frac{B}{C} \in \mathcal{T}$ and hence $B_1 \subseteq A$ because $\frac{B}{A} \in \mathcal{T}$. Thus A is $\mathcal{T}$ $-$essential.

Conversely, if A is $\mathcal{T}$ $-$pure and $\mathcal{T}$ $-$essential in B, if $\frac{B}{A}$ has any torsion submodule C then $C \approx B_1 \subseteq B$ and $A \cap B_1 = \{0\}$ for some $B_1 \subseteq B$ and $B_1 \in \mathcal{T}$, thus A cannot be $\mathcal{T}$ $-$essential. Hence, $\frac{B}{A} \in \mathcal{T}$.

Proposition1.13: The exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T}$ $-$pure exact if and only if $0 \longrightarrow \sigma(A) \longrightarrow \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ is a split exact sequence where the maps are restrictions of the above sequence.

Proof: Suppose that the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T}$ $-$pure exact. Now we complete the diagram by taking pullback of $j_C\colon \sigma(C) \longrightarrow C$ and $\pi\colon B \longrightarrow C$. Here, $:K \longrightarrow \sigma(B); u\colon \sigma(A) \longrightarrow \sigma(B);\quad v\colon \sigma(B) \longrightarrow \sigma(C);\ \alpha\colon \sigma(C) \rightarrow \sigma(B);\ s\colon \sigma(B) \rightarrow P.$

$$\begin{tikzcd}
 & & K \arrow[d] & & & \\
0 \arrow[r] & \sigma(A) \arrow[r] \arrow[d] & \sigma(B) \arrow[r] \arrow[d] & \sigma(C) \arrow[r] \arrow[d] & 0 & \ldots\ldots\ldots\ldots(1) \\
0 \arrow[r] & A \arrow[r] \arrow[d] & P \arrow[r, "\lambda"'] \arrow[d] & \sigma(C) \arrow[r] \arrow[d] & 0 & \ldots\ldots\ldots\ldots(2) \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 & \ldots\ldots\ldots(3)
\end{tikzcd}$$

$q\colon P \rightarrow B;\ j_B\colon \sigma(B) \longrightarrow B, i'\colon A \rightarrow P, \pi'\colon P \rightarrow \sigma(C);\ \lambda\colon \sigma(C) \rightarrow P, i\colon A \rightarrow B, \pi\colon B \rightarrow C$ are the required homomorphism. Here $s\colon \sigma(B) \rightarrow P$ exists as P is a pullback. Put $K = \ker(v)$. Now $vou = 0$ and so, $\sigma(A) \subseteq K$. since sequence (1) is $\mathcal{T}$ $-$pure $\Longrightarrow$ sequence (2) is $\mathcal{T}$ $-$pure because $\mathcal{T}$ $-$pure sequences form a proper class and hence (2) splits. Take $\lambda\colon \sigma(C) \rightarrow P$ such that $\pi' o\, \lambda = 1_{\sigma(C)}$. Now $\lambda(\sigma(C))$ is torsion and so there is $\alpha\colon \sigma(C) \rightarrow \sigma(B)$ such that $\lambda = so\alpha$. Also, $vo\alpha = \pi' o(so\alpha) = \pi' o\, \lambda = 1_{\sigma(C)}$ and hence, v is epic and

$0 \longrightarrow K \rightleftarrows \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ splits. But then K is an epimorphic image of $\sigma(B)$ and so, it is torsion. Also, $\pi' o(sot) = 0 \Longrightarrow K \subseteq A$. Hence, $K \subseteq \sigma(A)$ and sequence (1) is split and exact.

Conversely, if sequence (1) is split and exact, then given $T \in \mathcal{T}$, and $f\colon T \longrightarrow C, Im(f) \subseteq \sigma(C)$ and also, sequence (3) $\mathcal{T}$ $-$pure.

$$\begin{tikzcd}
 & & T \arrow[d] & & \\
0 \arrow[r] & \sigma(A) \arrow[r] \arrow[d] & \sigma(B) \arrow[r, shift left] \arrow[d] & \sigma(C) \arrow[l, shift left] \arrow[r] \arrow[d] & 0 \ldots\ldots\ldots\ldots(4) \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \ldots\ldots\ldots(5)
\end{tikzcd}$$

Note: If sequence (3) is $\mathcal{T}$ −pure, so it is exact on sequence (3) and hence, $\sigma(A) = A \cap \sigma(B)$ and $\frac{\sigma(B)+A}{A} = \sigma\left(\frac{B}{A}\right)$.

II. $\mathcal{T}_C$ − PURITY OF CYCLIC TORSION MODULES

Now we, define $\mathcal{T}_C -$ purity corresponding to the class $\mathcal{T}_C$ of cyclic torsion modules. This purity was firstly studied by **Stenstrom [17].** More generally he started with a family ϑ of factors of a projective generator $F$and he considered the purity $\pi^{-1}(\vartheta)$. Then he took the family $\vartheta' = \vartheta\backslash\{F\}$, and considered the relation between $\pi^{-1}(\vartheta)$ and the torsion theory generated by (ϑ'). He also proved that:

Theorem2.1: (i). $\sigma(M)$, the torsion submodule corresponding to the above torsion theory is the smallest ϑ −pure subobject of M such that each $f: P \longrightarrow M$ with $P \in \vartheta'$ factors through it.
(ii). $L \subseteq M$ is $\vartheta -$ pure in M and contains $\sigma(M)$ if and only if $\sigma\left(\frac{M}{L}\right) = 0$ for all $P \in \vartheta'$.

These conditions completely fulfill the case for $\mathcal{T}_C -$ purity, if given the torsion theory $(\mathcal{T}, \mathcal{T}_1)$, we take $\vartheta = \{R\} \cup \{all\ cyclic\ torsion\ modules\}$. Then the purity is $\pi^{-1}(\mathcal{T}_C)$.

Since, $\pi^{-1}(\mathcal{T}_C) = \pi^{-1}(\{R\} \cup \mathcal{T}_C)$ as R is projective and the generated torsion theory is same as the original.

We denote the set of dense left ideals by D that is there is left ideal I such that $R/I \in \mathcal{T}_C$. In this case $(\mathcal{T}, \mathcal{T}_1)$ is hereditary D forms a Gabriel filter or a topology **(Stenstrom [17]).**

$\mathcal{T}_C -$ purity coincides with the purity defined by **Lambek [16]** in case the torsion theory is hereditary.

Proposition2.2: If K is $\vartheta -$ pure in M, where ϑ is the class of all cyclic, if and only if given $\overline{m} \in \frac{M}{K}$, there exists, $m' \in M$ such that $(m - m') \in K$ and $Ann(m') = Ann(\overline{m})$.

Proposition2.3: If $(\mathcal{T}, \mathcal{T}_1)$ is hereditary then the following conditions are equivalent for a sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R -$ modules.

- A is $\mathcal{T}_C -$ pure in B.
- Given $n \in \sigma(C)$, there is $m \in B$ such that $Ann(m) = Ann(n)$ and $\lambda(m) = n$.
- A is pure in B in the sense of **Lambek [16]** that is given $m \in B$, and $D \in \mathcal{D}$ such that $Dm \subseteq A$, there is $l \in A$ such that $D(m - l) = 0$.

Proof: The proof of this theorem is analogous to the above proposition.

Note: For the case of abelian groups and the usual torsion theory, the above purity coincides with the usual purity.

Proposition 2.4: For any class ϑ, the following statements are equivalent for any $R -$ module M:

- (i). M is injective module relative to any sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R -$ modules with $C \in \vartheta$.
- (ii). M is absolutely $\vartheta -$ pure.
- (iii). $Ext(C, M) = 0$ for all $C \in \vartheta$.
- (iv). C is $i^{-1}(M) -$ flat for all $C \in \vartheta$.

Proof: $(ii) \Rightarrow (i)$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & P \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Since, it is given a homomorphism $A \rightarrow M$, we complete the diagram by pushout. Now (i) is $\vartheta -$ pure and hence homotopy exists, so M is injective relative to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R -$ modules with $C \in \vartheta$.
$(i) \Rightarrow (iii)$. Given any sequence : $0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0$, in which M is injective relative to it and hence it splits and so, $Ext(C, M) = 0$ for all $C \in \vartheta$.

$(iii) \Rightarrow (ii)$.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & A \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

It is given that E and $C \in \vartheta$, now we complete the above diagram by pullback. Now by the hypothesis of the upper sequence splits and hence there is a homotopy and hence, the given sequence is $\vartheta -$ pure. Therefore, M is absolutely $\vartheta -$ pure.

$(i) \Leftrightarrow (iv)$. It is obvious. That is M is injective module relative to any sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R -$ modules with $C \in \vartheta \Leftrightarrow C$ is $i^{-1}(M) -$ flat for all $C \in \vartheta$

Now dually we have M is $\vartheta -$ copure flat if and only if M is projective with respect to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $A \in \vartheta$ that is if and only if $Ext(M, A) = 0$ for all $A \in \vartheta$.

Now we try to give the relation between $\mathcal{T}$ −pure injective and $\mathcal{T}$ −pure projective modules.

Proposition 2.5: The following statements are equivalent for any $R-$ module M:
(i). M is $\mathcal{T}$ −pure injective.
(ii). M is injective with respect to closed sub-modules.
(iii). M is absolutely $\mathcal{T}_1-$ pure.
(iv). $Ext(F, M) = 0$ for all $F \in \mathcal{T}_1$.
(v). $Pext_{\mathcal{T}}(\mathrm{N, M}) = 0$ for all R −modules N.

Proof: $(i) \Rightarrow (v)$ Given any $\mathcal{T}$ −pure sequence $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$, as M is $\mathcal{T}$ −pure injective and so it splits. Hence, $Pext_{\mathcal{T}}(\mathrm{N, M}) = 0$ for all R −modules N.

$(v) \Rightarrow (i)$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \ldots\ldots\ldots\ldots(1) \\
0 \arrow[r] & M \arrow[r, shift left] & P \arrow[l, shift left] \arrow[r, shift left] & C \arrow[l, shift left] \arrow[r] & 0
\end{tikzcd}$$

Suppose that sequence (1) is a $\mathcal{T}$ −pure sequence and $f: A \longrightarrow M$ is given. Now we take a pushout, the lower sequence splits and hence M is $\mathcal{T}$ −pure injective.

$(v) \Rightarrow (iv)$. This statement follows because $Ext(F, M) = Pext_{\mathcal{T}}(F, M)$ for all $F \in \mathcal{T}_1$. Since, we know that $Pext_{\mathcal{T}}(F, M) = Ext(F, M)$ for all $F \in \mathcal{T}_1$.
$(iv) \Rightarrow (v)$. Conversely, if $Ext(F, M) = 0$ then the sequence $0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$ splits for all $F \in \mathcal{T}_1$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & \sigma(N) \arrow[rr, phantom, "="] \arrow[d] & & \sigma(N) \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[rr] \arrow[d] & & N \arrow[r] \arrow[d] & 0 \ldots\ldots\ldots(1) \\
0 \arrow[r] & K \arrow[r, shift left] & P/\sigma(N) \arrow[l, shift left] \arrow[rr] \arrow[d] & & N/\sigma(N) \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd}$$

Where,

$u: M \longrightarrow P, \pi: P \longrightarrow N, \lambda: N \longrightarrow \dfrac{N}{\sigma(N)}, \lambda': P \longrightarrow \dfrac{P}{\sigma(N)}, \mu: M \longrightarrow K,$

$\mu': K \longrightarrow M. i: \sigma(N) \longrightarrow N, j: \sigma(N) \longrightarrow P$ and $q: P/\sigma(N) \longrightarrow K$ are the maps.

Since, it is given that the sequence (1) is $\mathcal{T}$ −pure and we have $j: \sigma(N) \longrightarrow P$ which is a mono morphism. Now, $N/\sigma(N) \in \mathcal{T}_1$ and hence, the right vertical sequence $0 \longrightarrow \sigma(N) \longrightarrow N \longrightarrow N/\sigma(N) \longrightarrow 0$ is $\mathcal{T}$ −pure and hence, $\pi' \in \pi^{-1}(\mathcal{T})$ and so the epimorphism π' splits.

Now we considering the vertical exact sequence, the identity map above guarantees that the square is a pullback which in turn guarantees that μ is an isomorphism.
If $\mu' = \mu^{-1}$, then $\mu' o(q o \lambda') o\, u = (\mu' o q) o(u' o\, \mu) = \mu' o \mu = 1$ and hence, the upper sequence splits also and so, $Pext_{\mathcal{T}}(\mathrm{N, M}) = 0$ for all R −modules N.

$(iv) \Leftrightarrow (iii) \Leftrightarrow (ii)$. Follows from the previous proposition **(2.4)** by taking $\vartheta = \mathcal{T}_1$.

Note: $\mathcal{T}_1-$ purity arises in the theory of torsion free covers **(Teply and Golan [18]).**

Proposition2.6: If for any module $M, M/\sigma(M)$ is projective, then M is $\mathcal{T}$ −pure projective. Conversely, for every $\mathcal{T}$ −pure projective module M, $M/\sigma(M)$ is a projective module provided that every torsion free module is a factor of a projective torsion free module.

Conclusion: In this paper we consider an another notion of purity stronger than the **Cohn**'s **purity [11]**. If $\boldsymbol{FG}$ denotes the class of all finitely generated $\boldsymbol{R}$ −modules, since, this class is closed under factors. We shall try to give some characterizations of FG −purity and to determine its relationship with the $\mathcal{T}-$ purity and $\mathcal{T}_C-$ purity in cyclic torsion modules also derive some relations it with absolutely $\vartheta-$ pure modules. We try to relate it with **Teply and Golan [18]) conditions for** $\mathcal{T}-$ pure projectivity.

We shall try to give some characterizations of $\boldsymbol{FG}$ −purity and to determine its relationship with the $\boldsymbol{FG}$ −flat modules. We relativize this concept and also relate it with that of finite projectivity of **Azumaya** [8] with respect to a torsion theory and to study the inter-relationship between these concepts. We also consider finite $\boldsymbol{\sigma}$ −projectivity or $(\boldsymbol{FG}, \boldsymbol{\sigma})-$ pure flatness, cyclically $\boldsymbol{\sigma}-$ pure projectivity and cyclically $\boldsymbol{\sigma}-$ pure flatness.

III. CONCLUSION

In this paper we consider another notion of purity stronger than the Cohn's purity [11]. If denotes the class of all finitely generated modules, since, this class is closed under factors. We shall try to give some characterizations of purity and to determine its relationship with the purity and purityin cyclic torsion modules also derive some relations it with absolutely pure modules. We try to relate it with Teply and Golan [18]) conditions for pure projectivity.

We shall try to give some characterizations of purity and to determine its relationship with the flat modules. We relativize this concept and also relate it with that of finite projectivity of Azumaya[8] with respect to a torsion theory and to study the inter-relationship between these concepts. We also consider finite projectivity or pure flatness, cyclically pure projectivity and cyclically pure flatness.

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