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Birth and Death Process under the Influence of Catastrophes

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Abstract- Birth and death process have been studied very extensively in the past (see kendall (1948), bartlett (1955), feller (1957), harris (1963) and bailey (1964)). recently such processes have been studied allowing disasters to occur randomly over time decrementing the population size (see brockwell et al (1982), pakes (1987), bartoszynski et al(1989), buhelr and puri (1989) and peng et al (1993)). the motivation to study these processes stem from the fact several biological populations (for example, ungulate populations on sub-artic islands and populations of grizzy bears in yellowstone park) exhibit this type of behaviour (for detailed account of such examples, see hanson and trckwell (1987)). catastrophes are instantaneous events, each killing some of the members of the population who are present at the time of occurrence of the disaster.

Keywords- Feedback, Multiple Vacations, Starting Failures, Retrial Quene, Steady State Solution.

I. INTRODUCTION

In the Modelling of biological populations with birth-and-death process allowing disasters, one important aspect is the equal response Y(t) at any given instant of time which is defined by

$$Y(t)=\int_0^t X(u)\,du, \quad (1.1)$$

where X(t) denotes the population size at any time t. Puri(1966a,b) has initiated the interest for the study of the integral (2.1) by using linear birth-and-death processes in his analysis of the response of an animal after infection in the absence of defense mechanism. However, stochastic integral of the type (2.1) have been analyzedand the population process X(t) is a birth-and-death process which allows disasters. In this Chapter, an attempt is made to present some results pertaining to Y(t) when X(t) is a linear birth-and-death process under the influence of disasters.

II. MATHEMATICAL DESCRIPTION OF THE MODEL

Let X(t) of a population subject to randomly occurring disasters and also the quantal response Y(t) in connection with X(t).

1. The birth-and-death process $X(t)$ and the integral $Y(t)$

We consider the continuous-time evolution of a population initiated by an individual at time t = 0 and disturbed by disasters occurring as a Poisson stream with constant intensity β.

Let $T_{1,}T_2+T_{2,...}$be the time at which the successive disasters occur. Then $T_1, T_2, \ldots$are independent random variables having the common probability density function given by

$$f(t)=\beta e^{-\beta t}, \quad t>0 \qquad (2.2)$$

when the i^{th}disaster occurs, it affects all members present independently, killing an individual with constant probability 1-δ, leaving it unaffected with probability δ, where $\delta \in (0,1)$. Denoting the time of occurrence of the ith disasters as T_i, we observe that

$$T_i = T_1 + T_2 + \ldots + T_i \qquad (2.3)$$

and X (T_i+0) follows a binomial distribution with parameters $X(T_i-0)$? and δ. We assume that before T_i and in between any two successive disasters, the population evolves as time-homogeneous linear birth-and-death process with intensities $\lambda\, and\, \mu.$ Then the stochastic process $\{X(t), t \geq 0\}$ is called a birth-and-death process allowing disasters. We note that the process $X(t)$ is a special case of that introduced by Kaplan et al (1975) and further developed by Athreya and Kaplan (1976). For the process $X(t)$, we consider the cumulative process $Y(t)$ defined by (2.1). The joint-moment generating function of the vector process $(X(t), Y(t))$ is defined by

$$\psi(s_1, s_2, t) = E\left\{s_1^{X(t)} e^{-s_{2Y(t)}/} X(0) = 1\right\}(2.4)$$

We proceed to derive an integral equation satisfied by $\psi(s_1, s_2, t)$. We denote by ξ the event of a birth or a death or a disaster associated with the process $X(t)$. We first note that either no or atleast one ξ occurs in the interval $(0, t)$. We have the following mutually exclusive and exhaustive cases:

(i) No ξ occurs in $(0, t)$.

(ii) The first ξ in$(0,t)$ is a birth.
(iii) The firstξ in $(0,t)$ is a death.
(iv) The first ξ in $(0,t)$ is a disaster and it kills the ancestor.
(v) The first ξ in $(0,t)$ is a disaster and the ancestor survives.

We remark here that there is an alternative way of obtaining an integral equation using FourieSeries for $\psi(s_1,s_2,t)$ which utilizes the regenerative property at the time of occurrence of the first disaster. Now, conditioning on the time of occurrence of the first disaster and using probabilistic arguments, we obtain the following integral equation for $\psi(s_1,s_2,t)$:

$$\psi(s_1,s_2,t) = e^{-\beta t}\psi_0(s_1,s_2,t)\beta\int_0^t e^{-\beta\tau}\psi_0(1,s_2,\tau)\psi_0(1-\delta + \delta\psi(s_1,s_2,t,-\tau),0,\tau)d\tau$$

where $\psi_0(s_1,s_2,t)$ represents the joint-moment generating function of the vector process $(X_0(t),Y_0(t))$ disallowing disasters, defined by

$$\psi(s_1,s_2,t) = E\left\{s_1^{X_0(t)}e^{-s_2Y_0(t)}/X(0)=1\right\}(2.6)$$

Puri (1966) has explicitly obtained that

$$\psi_0(s_1,s_2,t) = r_2 + \frac{r_1-r_2}{1-\{(s_1-r_1)/(s_1-r_2)\}e^{\lambda(r_1-r_2)t}}(2.7)$$

where r_1 and r_2 are the roots of

$$\xi^2 + \frac{1}{\lambda}(s_2-\mu-\lambda)\xi + \frac{\mu}{\lambda} = 0 \qquad (2.8)$$

2.2 The Moments of the Process $\{X(t),Y(t)\}$

We use the following notation:

$$M_X(t) = E(X(t)), M_{X_0}(t) = E(X_0(t))$$
$$M_Y(t) = E(Y(t)), M_{Y_0}(t) = E(Y_0(t))$$
$$M_X^{(2)}(t) = E[(X(t)(X(t)-1))], M_{X_0}^{(2)}(t) = E[X_0(t)(X_0(t)-1)]$$
$$M_Y^{(2)}(t) = E(Y^2(t)), M_{Y_0}^{(2)}(t) = E[Y_0^2(t)]$$
$$M_{XY}^{(2)}(t) = E\left(X(t)(Y(t))\right), M_{X_0Y_0}^{(2)}(t) = E[X_0(t)(Y_0(t)]$$

Differentiating (2.5)with respect to s_1 at $(s_1=1,s_2=0)$, we get the following integral equation for the mean $M_X(t)$.

$$M_X(t) = e^{-(\lambda+\mu+\beta)t} + \int_0^t e^{-(\lambda+\mu+\beta)t}(2\lambda+\beta\delta)M_X(t-\tau)d\tau \qquad (2.9)$$

Differentiating (2.5) with respect to s_2 at $(s_1=1,s_2=0)$, we get the following integral equation for the mean $M_Y(t)$.

$$M_Y(t) = te^{-(\lambda+\mu+\beta)t} + \int_0^t e^{-(\lambda+\mu+\beta)t}\{(\lambda+\mu+\beta)\tau + (2\lambda+\beta\delta)\}M_X(t-\tau)\}d\tau \quad (2.10)$$

Differentiating (2.5) twice with respect to s_1 at $(s_1=1,s_2=0)$, we get the following integral equation for $M_X^{(2)}(t)$.

$$M_X^{(2)}(t) = \int_0^t e^{-(\lambda+\mu+\beta)\tau}\left\{2\lambda(M_X(t-\tau))^2 + (2\lambda+\beta\delta)M_X^{(2)}(t-\tau)\right\}d\tau \quad (2.11)$$

Differentiating (2.5) twice with respect to s_2 at $(s_1=1,s_2=0)$, we get

$M_Y^{(2)}(t) = t^2e^{-(\lambda+\mu+\beta)t} + \int_0^t e^{-(\lambda+\mu+\beta)t}\{(\lambda+\mu+$ $\beta)(\tau)2+22\lambda+\beta\delta\tau MYt-\tau+$ $2\lambda(MYt-\tau)2+(2\lambda+\beta\delta)MY2(t-\tau)\}d\tau$.(2.12)

Differentiating (2.5) successively with respect to $s_1$and s_2 at $(s_1=1,s_2=0)$, we get

$$M_{XY}^{(2)}(t) = te^{-(\lambda+\mu+\beta)t} + \int_0^t e^{-(\lambda+\mu+\beta)t}\{((2\lambda+\beta\delta)\tau M_X(t-\tau) + 2\lambda(M_X(t-\tau)M_Y(t-\tau)) + (2\lambda+\beta\delta)M_{XY}^{(2)}(t-\tau)\}d\tau(2.13)$$

(2.5)

The equations (2.9) to (2.13) are of renewal type and hence by applying the usual Laplace transform technique, the equations can be solved starting from (2.9) to get the various moments listed above. In what follows, $\rho=\lambda-\mu$ and the laplace transform of $f(t)$ is denoted by $f^*(\eta)$.

Taking Laplace transforms of the equations (2.9) to (2.13) and then simplifying, we get

$$M_X^*(\eta) = \frac{1}{\eta+\beta(1-\delta)-\rho},(2.14)$$

$$M_Y^*(\eta) = \frac{1}{\eta(\eta+\beta(1-\delta)-\rho)},(2.15)$$

$$M_X^{(2)*}(\eta) = \frac{2\lambda}{(\eta-\rho+\beta(1-\delta))(\eta-2\rho+2\beta(1-\delta))}, \quad (2.16)$$

$$M_Y^{(2)*}(\eta) = \frac{4\lambda}{\eta(\eta-\rho+\beta(1-\delta))^2(\eta-2\rho+2\beta(1-\delta))} + \frac{2}{\eta(\eta-\rho+\beta(1-\delta))^2},(2.17)$$

$$M_{XY}^{(2)*}(\eta) = \frac{\eta+\lambda+\mu+\beta(1-\delta)}{(\eta-\rho+\beta(1-\delta))^2(\eta-2\rho+2\beta(1-\delta))}, \quad (2.18)$$

which on inversion yield

$$M_X(t) = e^{(\rho-\beta(1-\delta))t},(2.19)$$

$$M_Y(t) = \frac{e^{(\rho-\beta(1-\delta))t}-1}{\rho-\beta(1-\delta)}(2.20)$$

$$M_X^{(2)}(t) = \frac{2}{\rho-\beta(1-\delta)}\{e^{2(\rho-\beta(1-\delta))t} - e^{(\rho-\beta(1-\delta))t}\}, \quad (2.21)$$

$$M_Y^{(2)}(t) = \frac{2\lambda}{(\rho-\beta(1-\delta))^3}\left(e^{2(\rho-\beta(1-\delta))t}-1\right) - 2\frac{(\lambda+\mu+\beta(1-\delta))}{(\rho-\beta(1-\delta))^2}te^{(\rho-\beta(1-\delta))t} - \frac{2}{(\rho-\beta(1-\delta))2}\left(e^{(\rho-\beta(1-\delta))t}-1\right), \quad (2.22)$$

$$M_{XY}^{(2)}(t) = e^{(\rho-\beta(1-\delta))t}\left\{\frac{2\lambda}{(\rho-\beta(1-\delta))^2}\left(e^{(\rho-\beta(1-\delta))t}-1\right) - t\frac{\lambda+\mu+\beta(1-\delta)}{\rho-\beta(1-\delta)}\right\} \quad (2.23)$$

Consequently, the variances of $X(t)\&Y(t)$ and the covariance of $X(t)\&\,Y(t)$ are given by

$$\text{Var}\left(X(t)\right) = \frac{\lambda+\mu+\beta(1-\delta)}{\lambda-\mu-\beta(1-\delta)}\left\{e^{2(\rho-\beta(1-\delta))t} - e^{(\rho-\beta(1-\delta))t}\right\}, \quad (2.24)$$

$$\text{Var}\left(Y(t)\right) = \frac{(\lambda+\mu+\beta(1-\delta)}{(\lambda-\mu-\beta(1-\delta))^3}\left\{e^{2(\rho-\beta(1-\delta))t}-1\right\} - 2t\frac{\lambda+\mu+\beta(1-\delta)}{(\lambda-\mu-\beta(1-\delta))^2} \quad (2.25)$$

$$cov\left(X(t),Y(t)\right) = \frac{\lambda+\mu+\beta(1-\delta)}{\lambda-\mu-\beta(1-\delta)}e^{(\rho-\beta(1-\delta))t}\left\{\frac{1}{\lambda-\mu-\beta(1-\delta)}\left(e^{(\rho-\beta(1-\delta))t}-1\right) - t \quad (2.26)$$

We desire to note that the integral (2.5) can be transformed into an equivalent differential equation given by

$$\frac{\partial\psi}{\partial t} = \lambda\psi^2 - (\lambda+\mu+\beta(1-\delta)+s_2)\psi + \mu + \beta(1-\delta) \quad (2.27)$$

The equation (2.7) can readily be solved and we obtain

$$\psi(s_1,s_2,t) = r_2 + \frac{r_1-r_2}{1-\frac{s_1-r_1}{s_2-r_2}e^{\lambda(r_1-r_2)t}} \quad (2.28)$$

where r_1 *and* r_2 are the roots of the quadratic equation

$$\xi^2 - \frac{1}{\lambda}(\lambda+\mu+\beta(1-\delta)+s_2)\xi + \frac{1}{\lambda}\left(\mu+\beta(1-\delta)\right) = 0 \quad (2.29)$$

III.CONCLUSION

These catastrophes may occur at any time of instant and leads to annihilation of all customers in the system. Catastrophe makes a system to be inactive. We have obtained the time dependent solution of our model. The busy period analysis has been carried out and the results was used in the study steady state solution of the queuing model. We have the probability generating function of transient solutions explicitly and along with this the steady state has also being analyzed.

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