

A Study on Various Continuous Functions

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Abstract- In this paper, we present and study a new concepts namely strongly rb-continuous and Perfectly rb-continuous, Contra rb-continuous and Totally rb-continuous. Also examine some of their properties of such functions.

Keywords- rb*-continuous, Perfectly rb-continuous, Contra rb-continuous, Totally rb-continuous.

I. INTRODUCTION

The concept of functions is one of the most essential concepts in modern mathematics. Over the years, different forms of functions have been presented and various interesting problems arise out of it. Thus, the study on this conception has developed and evolved continuously up to the current times. Levin [3] introduced strongly continuous maps, and Noiri [5] introduced and studied perfectly continuous maps. Dontchev et al. initially proposed the concept of contra-continuity in 1994 [1]. RC Jain [2] first developed the ideas of totally continuous functions and slightly continuous for topological spaces. The concept of rb-continuous functions was created by Nagaveni et al. [4].

Throughout this paper (X, τ) , (Y, σ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure and interior of A with respect to τ are denoted by $cl(A)$ and $int(A)$ respectively.

II. PRELIMINARIES

First we recall the following definitions which are used in our study.

1. Definition 2.1

- A map $f : X \rightarrow Y$ is called continuous if $f^{-1}(V)$ is closed in (X, τ) for every closed set V of (Y, σ) .
- A map $f : X \rightarrow Y$ is called strongly continuous if $f^{-1}(V)$ is both open and closed in (X, τ) for every subset V of (Y, σ) .
- A map $f : X \rightarrow Y$ is called perfectly continuous if $f^{-1}(V)$ is both open and closed in (X, τ) for every closed subset V of (Y, σ) .
- A map $f : X \rightarrow Y$ is called totally continuous if the inverse image of every closed subset of (Y, σ) is a clopen subset of (X, τ) .
- A map $f : X \rightarrow Y$ is called contra continuous if the inverse image of every closed subset of (Y, σ) is an open subset of (X, τ) .

- A map $f : X \rightarrow Y$ is called continuous if $f^{-1}(V)$ is rb-closed in (X, τ) for every closed set V of (Y, σ) .

III. RB-CONTINUOUS FUNCTION IN TOPOLOGICAL SPACES

Theorem 3.1:

Every rb-continuous function is rg -continuous function but not conversely.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb-continuous function. Let V be any closed set in Y . f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ) . It follows from the fact that every rb-closed set is rg -closed. Hence $f^{-1}(V)$ is rg -closed in (X, τ) . Therefore, f is rg -continuous.

Converse need not be true as seen from the following example

Example 3.2:

Let

$$X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{X, \emptyset, \{c\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by identity mapping it is clear that f is continuous function but not rb-continuous function. Since $\{a, b\}$ is closed in (Y, σ) , $f^{-1}(\{a, b\}) = \{a, b\}$ is not rb-closed in (X, τ) .

Theorem 3.3: Every rb-continuous function is αg -continuous function but not conversely.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb-continuous function. Let V be any closed set in Y . f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ) . Then by thore [14] rb-closed set is αg -closed. Hence $f^{-1}(V)$ is αg -closed in (X, τ) . Therefore, f is αg -continuous.

Converse need not be true as seen from the following example.

Example 3.4: Let $X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}, \sigma = \{X, \emptyset, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function

defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is αg -continuous function but not rb -continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not rb -closed in (X, τ) .

Theorem 3.5: Every rb -continuous function is $g\alpha$ -continuous function but not conversely.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb -continuous function. Let V be any closed set in Y . f is rb -continuous function $f^{-1}(V)$ is rb -closed in (X, τ) . Then by [14] rb -closed set is $g\alpha$ -closed. Hence $f^{-1}(V)$ is $g\alpha$ -closed in (X, τ) . Therefore f is continuous.

Converse need not be true as seen from the following example

Example 3.6:
Let

$$X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}, \sigma = \{X, \emptyset, \{c\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by identity mapping it is clear that f is $g\alpha$ -continuous function but not rb -continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not rb -closed in (X, τ) .

Theorem 3.7: Every rb -continuous function is pre-continuous function but not conversely.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb -continuous function. Let V be any closed set in Y . f is rb -continuous function $f^{-1}(V)$ is rb -closed in (X, τ) . Then by [14] rb -closed set is pre-closed. Hence $f^{-1}(V)$ is pre-closed in (X, τ) . Therefore f is pre-continuous.

Converse need not be true as seen from the following example

Example 3.8:
Let

$$X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}, \sigma = \{X, \emptyset, \{a, b\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by identity mapping it is clear that f is pre-continuous function but not rb -continuous function. Since $\{c\}$ is pre-closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not rb -closed in (X, τ) .

Theorem 3.9: Every rb -continuous function is gb -continuous function.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb -continuous function. Let V be any closed set in Y . f is rb -continuous function $f^{-1}(V)$ is rb -closed in (X, τ) . Then by [14] rb -closed set is gb -closed. Hence $f^{-1}(V)$ is gb -closed in (X, τ) . Hence, f is α -continuous.

Remark 3.10: Converse of the above theorem need not be true as seen from the following example.

Example 3.11:
Let

$$X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}, \sigma = \{X, \emptyset, \{a, b\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is gb -continuous function but not rb -continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is rb -closed in (X, τ) .

IV. STRONGLY REGULAR B – CONTINUOUS

Definition 4.1 A function $f : X \rightarrow Y$ from a topological space X into a topological space Y is said to be strongly regular b – continuous functions (briefly rb^* -continuous) if the inverse image of every rb -closed set in Y is closed in X .

Proposition 4.2: If a function $f : X \rightarrow Y$ is continuous then it is rb^* – continuous but not conversely.

Proof: Assume that f is continuous. Let G is any closed set in Y . Since every rb -closed set is closed, G is closed set in Y . Since f continuous, $f^{-1}(G)$ is closed in X . Therefore f is rb^* -continuous.

Converse need not be true as seen from the following example

Example 4.3: Let $X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}, \sigma = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be a function defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is rb^* -continuous function but not continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not closed in (X, τ) .

Proposition 4.4 A function $f : X \rightarrow Y$ is strongly rb – continuous if and only if the inverse image of every rb -open set in Y is closed in X .

Proof: Assume that f is rb^* – continuous. Let F be any rb -open set in Y . Then F^c is rb -closed in Y . since f is rb^* – continuous. $f^{-1}(F^c)$ is closed in X . But $f^{-1}(F^c) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is open in X . Converse assume that the inverse image of every rb -open set in Y is open in X . Let G be any rb -closed set in Y . then G^c is rb -open in Y . by assumption, $f^{-1}(G^c)$ is open in X . but $f^{-1}(G^c) = X - f^{-1}(G)$ and so $f^{-1}(G)$ is closed in X . Therefore f is rb^* -continuous.

Proposition 4.5 If a function $f : X \rightarrow Y$ is strongly continuous then it is rb^* – continuous but not conversely.

Proof: Assume that f is strongly continuous. Let G is any rb -closed set in Y . Since f strongly continuous, $f^{-1}(G)$ is closed in X . Therefore f is rb^* -continuous.

Converse need not be true as seen from the following example.

Example: 4.6

Let

$$X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{Y, \emptyset, \{a, b\}\}.$$

Let $f : X \rightarrow Y$ be a function defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is rb^* -continuous function but not strongly continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is closed in X but not open X .

Proposition 4.7

If a function $f : X \rightarrow Y$ from a topological space X into a topological space Y is rb^* – continuous and a function $g : Y \rightarrow Z$ is rb -continuous then the composition.

$$g \circ f : X \rightarrow z \text{ is continuous.}$$

Proof: Let G be any closed set in Z . since g is rb -continuous, $g^{-1}(G)$ is rb -closed in Y . since f is rb^* – continuous, $f^{-1}(g^{-1}(G))$ is closed in X . But $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$. Therefore, $g \circ f$ is continuous.

V. PERFECTLY REGULAR B – CONTINUOUS FUNCTIONS

Definition 5.1

If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a topological space X into a topological space Y is said to be perfectly regular b – continuous functions (briefly Prb -continuous) if the inverse image of every rb -closed set in y is both open and closed in X .

Proposition 5.2

If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a topological space X into a topological space Y is perfectly rb -continuous then it is rb^* -continuous but not conversely.

Proof:

Assume that f is perfectly rb -continuous. Let G be any rb -closed set in Y . since every rb -closed set is closed, G is closed set in y . Since f continuous, $f^{-1}(G)$ is closed in X . Therefore f is rb^* -continuous.

Converse need not be true as seen from the following example.

Example: 5.3

Let

$$X = Y = \{a, b, c\}, \\ \tau = \{X, \emptyset, \{a\}\}, \\ \sigma = \{X, \emptyset, \{a\}, \{a, b\}\}.$$

Let $f : X \rightarrow Y$ be a function defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is rb^* -continuous function but

not Prb - continuous function. Since $\{b, c\}$ is rb -closed in $Y, f^{-1}(\{b, c\}) = \{b, c\}$ is not open in X .

Proposition 5.4

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a topological space X into a topological space Y is perfectly rb -continuous if and only if the inverse image of every rb -closed set in Y is both open and closed in X

Proof:

Assume that f is perfectly rb -continuous. Let F be any rb -open set in y . Then F^c is rb -closed in y . Since f is perfectly rb -continuous. $f^{-1}(F^c)$ is both open and closed in X . But $f^{-1}(F^c) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is both open and closed in X .

Converse assume that the inverse image of every rb -open set in y is both open and closed in X . Let G be any rb -closed set in y . then G^c is rb -open in y . By assumption, $f^{-1}(G^c)$ is both open and closed in X . But $f^{-1}(G^c) = X - f^{-1}(G)$ and so $f^{-1}(G)$ is open in X . Therefore f is perfectly rb -continuous.

VI. CONTRA REGULAR B-CONTINUOUS FUNCTION

In this section we define the function called Contra rb - continuous function and study some of its properties.

Definition 6.1:

The map $f : (X, \tau) \rightarrow (Y, \sigma)$ is Contra rb -continuous, if the inverse image of every open set in Y is rb -closed set in X .

Example 6.2:

Let

$$X = \{1, 2, 3, 4, 5\}, \\ \tau = \{X, \emptyset, \{1, 5\}, \{1, 3, 4, 5\}, \{3, 4\}\}, \\ Y = \{1, 2, 3, 4, 5\}, \\ \sigma = \{Y, \emptyset, \{1, 5\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map.

Then f is contra rb –continuous. Since $\{1,5\}$ is open in $(Y, \sigma), f^{-1}(\{1,5\}) = \{1,5\}$ rb - closed in (X, τ) That is, the inverse image of every open set in Y is rb - closed in X . Therefore f is contra rb -continuous.

Theorem 6.3:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ is Contra rb -continuous function, if and only if inverse image of every Closed set in Y is rb -open in X .

Proof:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and \mathcal{P} be a closed set in Y . Since f is Contra rb -continuous function $f^{-1}(Y - \mathcal{P}) = X -$

$f^{-1}(\mathcal{P})$ is closed set in X . Therefore $f^{-1}(\mathcal{P})$ is rb -open set in Y .

Conversely, let \mathcal{P} be a open set in Y . By assumption is $f^{-1}(Y - \mathcal{P})$ rb -open set. $f^{-1}(Y - \mathcal{P}) = X - \phi^{-1}(\mathcal{P}), f^{-1}(\mathcal{P})$ is rb -closed set in X . Hence f is Contra rb -continuous function.

Theorem 6.4:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function from X to Y , then the following conditions are equivalent.

- (i) Inverse image of every Closed set in Y is rb -open in X
- (ii) For $x \in X$ and each closed set \mathcal{P} in Y with $f(x) \in \mathcal{P}$, then there exist $anrb$ -open set Q in X such that $f(Q) \subseteq \mathcal{P}$.

Proof:

$$(i) \rightarrow (ii)$$

Consider \mathcal{P} be Closed set in Y such that $f(x) \in \mathcal{P}, x \in X$. Let $Q = f^{-1}(\mathcal{P})$,

$$(ii) \rightarrow (i)$$

Let \mathcal{P} be any Closed in $Y, x \in X, f(x) \in \mathcal{P}$. There exists an rb -open set X such that $f(X) \subseteq \mathcal{P}, f^{-1}(\mathcal{P}) = \cup \{X, x \in f^{-1}(\mathcal{P}) \in \tau(x)\}, f^{-1}(\mathcal{P})$ is rb -open set in X .

Remark 6.5:

Contra rb -continuous and rb -continuous are independent.

Example 6.6:

Let

$$X = Y = \{1, 2, 3\}, \\ \tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}, \\ \sigma = \{Y, \emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is contra rb -continuous. Since the inverse image of every open set in Y is rb -closed in X . Therefore f is contra rb -continuous. But f is not rb continuous. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not rb -closed in (X, τ) .

Example 6.7:

Let

$$X = Y = \{1, 2, 3\}, \\ \tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}, \\ \sigma = \{Y, \emptyset, \{1, 3\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is rb -continuous. Since the inverse image of every closed set in Y is rb -closed in X . Therefore f is rb -continuous. But f is not contra rb continuous. Since $\{1, 3\}$ is open in $(Y, \sigma), f^{-1}(\{1, 3\}) = \{1, 3\}$ is not rb -closed in (X, τ) .

Theorem 6.8

Every contra - continuous function is contra rb -continuous.

Proof:

It follows from the fact that every rb -closed set is closed set. The converse of the above theorem is not true as seen from the following example.

Example 6.9:

Let

$$X = Y = \{1, 2, 3\}, \\ \tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}, \\ \sigma = \{Y, \emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is contra rb -continuous. Since the inverse image of every open set in Y is rb -closed in X . Therefore f is contra rb -continuous. But f is not rb continuous. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is not rb -closed in (X, τ) .

VII. TOTALLY RB-CONTINUOUS FUNCTION

Definition 7.1:

A map $f : X \rightarrow Y$ is called totally rb -continuous if the inverse image of every closed subset of (Y, σ) is a rb -clopen subset of (X, τ) .

Proposition 7.2:

A function $f : X \rightarrow Y$ is totally rb -continuous if and only if the inverse image of every closed subset in Y is rb -clopen in X .

Proof:

Assume that f is totally rb -continuous. Let F be any closed subset in Y . Then F^c is open subset in Y . Since f is totally rb -continuous. $f^{-1}(F^c)$ is rb -clopen subset in X . But $f^{-1}(F^c) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is both rb -closed and rb -open subset. Hence $f^{-1}(F)$ is rb -clopen subset in X .

Converse, Let G be any open subset in y . Then G^c is closed subset in X . By the assumption $f^{-1}(G^c)$ is rb -clopen subset in X . But $f^{-1}(G^c) = X - f^{-1}(G)$ and so $f^{-1}(G)$ is both rb -open and rb -closed subset. Hence $f^{-1}(G)$ is rb -clopen subset in X . Therefore, f is totally rb -continuous.

Proposition 7.3:

Every totally rb -continuous $f : X \rightarrow Y$ is a

- totally b -continuous function
- b -continuous function
- rb -continuous function
- contra rb -continuous function

Proof:

(i) Let A be an open subset in (y, σ) . Since f is totally rb -continuous function. Thus, $f^{-1}(A)$ is rb -clopen subset in

X. since every rb-open set is b-open and every rb-closed is b-closed, we get $f^{-1}(A)$ is both b- open and b-closed subset in X. this implies that $f^{-1}(A)$ is b-clopen subset in X. Therefore, f is totally b-continuous.

(ii) Let A be an open subset in (y, σ) . Since f is totally rb – continuous function. Thus, $f^{-1}(A)$ is rb-clopen subset in X. since every rb-open set is b-open and every rb-closed is b-closed, we get $f^{-1}(A)$ is both b- open and b-closed subset in X. thus implies that $f^{-1}(A)$ is b-open subset in X. Therefore, f is b continuous.

(iii) Let A be an open subset in (y, σ) . Since f is totally rb – continuous function. Thus, $f^{-1}(A)$ is rb-clopen subset in X. Thus $f^{-1}(A)$ is rb-open subset in X. Therefore, f is totally rb-continuous.

(iv) Let A be an open subset in (y, σ) . Since ϕ is totallyrb – continuous. Thus, $f^{-1}(A)$ is rb-clopen subset in X. Thus $f^{-1}(A)$ is rb-closed subset in X. Therefore f is contra rb-continuous.

Remark 7.4

Converse of the above theorem need not be true as seen from the following example.

Example 7.5:

Let

$$\begin{aligned} X &= Y = \{a, b, c\}, \\ \tau &= \{X, \emptyset, \{a\}, \{b, c\}\}, \\ \sigma &= \{X, \emptyset, \{a, b\}\}. \end{aligned}$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is totally b-continuous function but not totally rb- continuous function. Since $\{c\}$ is closed in $(Y, \sigma), f^{-1}(\{c\}) = \{c\}$ is b-closed in (X, τ) but not rb-closed.

Example 7.6:

Let

$$\begin{aligned} X &= Y = \{a, b, c\}, \\ \tau &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \\ \sigma &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}. \end{aligned}$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity mapping, it is clear that f is b-continuous but not totally rb- continuous function. Since $\{a, c\}$ is closed in $(Y, \sigma), f^{-1}(\{a, c\}) = \{a, c\}$ is rb-closed in (X, τ) but not rb-closed.

Example 7.7

Let

$$\begin{aligned} X &= Y = \{a, b, c\}, \\ \tau &= \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}, \\ \sigma &= \{X, \emptyset, \{a\}, \{a, c\}\} \end{aligned}$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by : $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is rb-continuous function but not totally rb- continuous function. Since $\{b, c\}$ is closed in $(Y, \sigma), f^{-1}(\{b, c\}) = \{b, c\}$ is rb-closed in (X, τ) but not rb-open.

Example 7.8

Let

$$\begin{aligned} X &= Y = \{a, b, c\}, \\ \tau &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \\ \sigma &= \{X, \emptyset, \{a, b\}\}. \end{aligned}$$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity mapping, it is clear that f is contra rb-continuous but not totally rb-continuous function. Since $\{a, b\}$ is open in $(Y, \sigma), f^{-1}(\{a, b\}) = \{a, b\}$ is rb-closed in (X, τ) but not rb-closed.

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