International Journal of Scientific Research & Engineering Trends
Volume 6, Issue 5, Sept-Oct-2020, ISSN (Online): 2395-566X

Implementation of a New Fourth Order Kutta's Formula for Solving Initial Value Problems in Ordinary Differential Equations

Ononogbo Chibuike Benjamin, Airemen Ikhuoria Edward, EzurikeUgochi Julia, Ugbolo Cletus
Department of Mathematics,
Ambrose Alli University, Nigeria

***Abstract*-In this paper, we modified Kutta's Formula for solving initial value problems (IVPs) in Ordinary Differential Equation to a geometric Mean other than the conventional Arithmetic mean. Due to the vital role played by the method in the field of computation and applied science/engineering, we simplify and further reduce the complexity of its derivation and analysis by exploring some possibly well-know works and proposed a step by step derivation of the method. The new Algorithm was further implemented and compared with other existing methods, and the results indicated that the new method is of high degree of accuracy in comparison with other existing methods, this shows that the new method can be use to solve real life problem, that can possibly be reduced to first order ordinary differential equations. Errors involved in the new method and other existing methods, were plotted with MATLAB to obtain their trajectories. We called this formula OCB4 method.**

***Keywords*-Initial value problem, Kutta's method, Geometric mean, Taylor's series,**

I.INTRODUCTION

Many problems in science and engineering when formulated mathematically are readily expressed in terms of linear or non linear ordinary differential equations with appropriate initial or boundary conditions [10]. For example the trajectory of a ballistic missile, the motion of an artificial satellite in its orbit and many others, can be governed by ordinary differential equations. In addition, theories concerning electrical networking, bending of beams, stability of aircraft etc., are modeled by differential equations. Unfortunately, many of these differential equations cannot be solved by analytical techniques. This is why numerical treatment is very important and provides a powerful alternative tool for solving the differential equations [1].

Among the existing numerical methods, Runge-kutta method is widely used computationally because of its efficiency in terms of accuracy. There exist several version of Runge-Kutta methods, namely second order Runge-Kutta , fourth order and so on. The classical fourth order Runge-Kutta method can be expressed as Arithmetic Mean (AM) based approach. In the last few years, there has been a growing interest in problem solving system based on the Runge-Kutta method. Several methods have been developed using the idea of different means such as centroidal mean, harmonic mean, contra-harmonic mean, heronian mean and geometric mean, but our interest will be on Geometric mean. According to [3], revealed that, these different approaches performed better than the usual Arithmetic Mean based approach in many cases and compare favorably well. [5] presented a 4thorder Runge-Kutta method using the Geometric mean based, [3] proposed a new 4th order hybride Runge-Kutta method based on Geometric Mean, [2] propose a third order Rung-Kutta method based on linear combination of Arithmetic mean, harmonic mean and Geometric Mean, [8] developed fourth stage Harmonic Explicit Runge-Kutta method. [9] developed a new third order Runge-Kutta based on contraharmonic mean for stiff problems. It is pertinent to mention that no effort, so far, has been made to modify Kutta's algorithm based on Geometric Mean. Keeping this in view, a modest effort has been made in the present paper to develop such a new efficient numerical algorithm which is, for the first time added to the literature. It is observed that the presently developed method has also been found to be more suitable one for the solution of the initial value problem of the type:

$$y' = f(x,y), y(x_0) = y_0, x \in [a,b] \text{ ------------(1)}$$

Where gradient function f(x,y) may have points of discontinuities and the specific objectives are to;

- Modify Kutta's formula to Geometric Mean other than the usual Arithmetic Mean for solving initial value problem.
- implement and compare the performance of the new method with existing methods using some tested initial value problems;
- draw the error trajectories, through the help of MATLAB software.

International Journal of Scientific Research & Engineering Trends
Volume 6, Issue 5, Sept-Oct-2020, ISSN (Online): 2395-566X

For the purpose of clarity, the following definitions are necessary.

1. Definition of terms

In this paper, we present the following definitions for better understanding. Consider the general one-step explicit Runge-Kutta method given by:

$$y_{n+1} - y_n = h\phi(x_n, y_n, h) \text{ ------------------(2)}$$

[10] gives the following definitions;

Definition 1: Order of differential equation: The general one-step method (1) is said to be of order P, if P is the largest integer for which

$$y(x+h) - y(x) - h\phi(x, y(x), h) = o(h^{p+1}) \text{ -------(3)}$$

holds, where y(x) is the theoretical solution of the initial value problem

Definition 2: Local Truncation Error: The local truncation error at $x_{n+1}$of the general explicit one-step method in (2) defined asT_{n+1},where

$$T_{n+1} = y(x_{n+1}) - y(x) - h\phi(x, y(x), h) \text{ -----(4)}$$

And y(x) is the theoretical solution of the initial value problem. This paper has the following structure: Section 2 presents derivation of OCB4 method. Section 3, presents the implementation in comparison with existing methods with tested examples, numerical results are presented in tables by way of comparison. Section 4 presents, the Error trajectory of the methods. Finally, section 5 presents the summary and conclusion.

II. DERIVATION OF OCB4 METHOD

The OCB4 method is derived from the existing 4th order Kutta's method which is based on the conventional Arithmetic mean. The general S- stage Runge-Kutta method for solving an IVP is given in equation (1) defined from the well-known numerical integrator as:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

Where

$$\phi(x_n, y_n, h) = \sum_{i=1}^{s} b_i k_i$$

With

$$k_i = f\left(x_n + c_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j\right) - - - - (5)$$

$$c_i = \sum_{j=1}^{i-j} a_{ij} \qquad and\ c_i \in [0,1]$$

$$k_1 = f(x_n y_n) \text{-----------------(5a)}$$

$$k_2 = f(x_n + hc_2, y_n + a_{21}k_1) \text{--------------(5b)}$$

$$k_3 = f(x_n + hc_3, y_n + h(a_{31}k_1 + a_{32}k_2) \text{--------(5c)}$$

$$k_4 = f(x_n + hc_4, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3) \text{---(5d)}$$

For the purpose of linearity, the above parameters will be modified as follows:

$$a_{21} = a_1,\ a_{31} = a_2,\ a_{32} = a_3,\ a_{41} = a_4,\ a_{42} = a_5,\ a_{43} = a_6$$

Substituting we have;

$$k_1 = f(y_n) \text{-----------(6a)}$$

$$k_2 = f\big(y_n + h(a_1k_1)\big) \text{ ---------------(6b)}$$

$$k_3 = f\big(y_n + h(a_2k_1 + a_3k_2)\big) \text{-------------(6c)}$$

$$k_4 = f\big(y_n + h(a_4k_1 + a_5k_2 + a_6k_3)\big) \text{------------(6d)}$$

Adopting Taylor series expansion about the point (yn) of equation (6a) to (6d) we have:

$$\left.\begin{aligned}
k_1 &= f(y_n) \\
k_2 &= \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + ha_1k_1 \frac{d}{dy}\right)^i f(y_n) \\
k_3 &= \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h(a_2k_1 + a_3k_2)\frac{d}{dy}\right)^i f(y_n) \\
k_4 &= \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h(a_4k_1 + a_5k_2 + a_6k_3)\frac{d}{dy}\right)^i f(y_n)
\end{aligned}\right\}$$

----------(7)

According to [10] the fourth order Kutta's method for solving IVP of equation (1) is given by:

$$y_{n+1} = y_n + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4) \text{-------(8)}$$

Where

$$k_1 = f(y_n) \text{ -----------------(8a)}$$

$$k_2 = f\left(y_n + \tfrac{1}{3}hk_1\right) \text{ ------------------(8b)}$$

$$k_3 = f\left(y_n - \tfrac{1}{3}hk_1 + hk_2\right) \text{------------------(8c)}$$

$$k_4 = f(y_n + hk_1 - hk_2 + hk_3) \text{------------------- (8d)}$$

The main arm is to modify the Kutta's method to have a geometric progression. Recall the process of arithmetic mean of arbitrary numbers p, q, r with their common difference of progression as $q - p = r - q$ then $q = \frac{r+p}{2}$ From this claim, it implies that equation (8) can be written as

$$y_{n+1} = y_n + \frac{h}{4}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_2}{2} + \frac{k_3+k_3}{2} + \frac{k_3+k_4}{2}\right) \text{-------(9)}$$

Equation (9) is known as Kutta's method based on arithmetic mean. Using the same approach, if the three arbitrary numbers p, q, r are in a geometric progression, q been the geometric mean of "p","r" , their common ratio is $\frac{q}{p} = \frac{r}{q} \Rightarrow q = \sqrt{pr}$ or p = $\sqrt{qr}$ or r = $\sqrt{pq}$ with this analogy equation (9) can be modified into a geometric mean as;

$y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1k_2} + \sqrt{k_2k_2} + \sqrt{k_3k_3} + \sqrt{k_3k_4}\right)$ ------------------(10a)

$\Rightarrow y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1k_2} + k_2 + k_3 + \sqrt{k_3k_4}\right)$-------------------------(10b)

with $k_1, k_2, k_3 and\ k_4$ defined as in (6a) to (6d) respectively.

This means that equation (1) can be redefined by setting:

$\sqrt{k_1k_2}$ = f $(1+x)^{\frac{1}{2}}$ -----------------(11a)

$k_1k_2 = f^2(1+x)$------------(11b)

From (11b) we have that;

$x = \left(\frac{k_1k_2}{f^2} - 1\right)$-------------(12)

To evaluate the RHS of (10b), we apply a binomial expansion technique with fractional index

$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$ -----------(13)

Substituting equation (12) into (13) we have;

$\sqrt{k_1k_2} = 1 + \frac{1}{2}\left(\frac{k_1k_2}{f^2} - 1\right) - \frac{1}{8}\left(\frac{k_1k_2}{f^2} - 1\right)^2 + \frac{1}{16}\left(\frac{k_1k_2}{f^2} - \right.$ *13+…* ---------------------(14)

Simplifying equation (14) we obtain;

$\sqrt{k_1k_2} = \frac{5}{16} + \frac{15}{16f^2}k_1k_2 - \frac{5}{16f^4}k_1^2k_2^2 + \frac{1}{16f^6}k_1^3k_2^3$ ---(15)

Next we obtain the Taylor expansion of equation (7) as;

$k_1 = f$----------(16)

$k_2 = k_1 + ha_1k_1f_y + \frac{1}{2}h^2a_1^2k_1^2f_{yy} + \frac{1}{6}h^3a_1^3k_1^3f_{yyy}$ --(17)

$k_1k_2 = k_1^2 + ha_1k_1^2f_y + \frac{1}{2}h^2a_1^2k_1^3f_{yy} + \frac{1}{6}h^3a_1^3k_1^4f_{yyy}$ -------------(18)

$k_1^2k_2^2 = k_1^4 + 2k_1^4ha_1f_y + h^2a_1^2f_yk_1^4 + h^2a_1^2f_{yy}k_1^5 + k_1^5h^3a_1^3f_yf_{yy} + \frac{1}{3}k_1^6h^3a_1^3f_{yyy}$ -----------------(19)

$k_1^2k_2^3 = k_1^6 + 3k_1^6ha_1f_y + \frac{3}{2}k_1^7h^2a_1^2f_{yy} + 3k_1^6h^2a_1^2f_y^2 + 3k_1^7h^3a_1^3f_yf_{yy} + k_1^6h^3a_1^3f_y^3 + \frac{1}{2}k_1^8h^3a_1^3f_{yyy}$ ----------(20)

Substituting (18) to (20) into (15) and simplifying we have;

$\sqrt{k_1k_2} = 1 + \frac{1}{2}ha_1f_y + \frac{1}{4}h^2a_1^2k_1f_{yy} + \frac{1}{12}h^3a_1^3k_1^2f_{yy} - \frac{1}{8}h^2a_1^2f_y^2 - \frac{1}{8}k_1h^3a_1^3f_yf_{yy} + \frac{1}{16}h^3a_1^3f_y^3$-----------(21)

Similarly, we obtain

$\sqrt{k_2k_2} = k_2 = 1 + ha_1k_1f_y + \frac{1}{2}h^2a_1^2k_1f_{yy} + \frac{1}{6}h^3a_1^3k_1^2f_{yy}$------------------------(22)

$$\sqrt{k_3k_3} = k_3 = 1 + k_1c_3f_yh + \frac{1}{2}h^2k_1^2c_3^2f_{yy} + h^2a_3a_1k_1f_y^2 + \frac{1}{2}h^3a_3a_1k_1^2(a_1 + 2c_3)f_{yy}f_y + \frac{1}{6}h^3k_1^3c_3^3f_{yyy} -----(23)$$

Finally, we expand

$\sqrt{k_3k_4} = 1 + \frac{1}{2}\left(\frac{k_3k_4}{f^2} - 1\right) - \frac{1}{8}\left(\frac{k_3k_4}{f^2} - 1\right)^2 + \frac{1}{16}\left(\frac{k_3k_4}{f^2} - \right.$ *13+…*----------------(24)

$\sqrt{k_3k_4} = \frac{5}{16} + \frac{15}{16f^2}k_3k_4 - \frac{5}{16f^4}k_3^2k_4^2 + \frac{1}{16f^6}k_3^3k_4^3$----(25)

In a similar way we search for the Taylor expansion of equation (7) in other to obtain$k_4, k_3k_4, k_3^2k_4^2$and $k_3^3k_4^3$ which was obtain as;

$k_4 = 1 + k_1c_4f_yh + h^2k_1(a_1a_5 + a_6c_3)f_y^2 + \frac{1}{2}h^2k_1^2c_4^2f_{yy} + h^3a_6a_3a_1k_1f_y^3 + \frac{1}{2}h^3k_1^2k_1(a_1^2a_5 +$

2a1a5c4+a6c32+2a6c3c4fyyfy+16h3k13c43fyyy----------------(26)

$k_3k_4 = 1 + \frac{1}{2}f_y(c_3 + c_4)h + \frac{1}{8}f_y^3(4a_1a_3 + 4a_1a_5 +$ *4a6c3−c32+2c3c4−c42h2+14fyyk1c32+c42h2+116h38a1a3a6−4a1a3c3+4a1a3c4+4a1a5c3−4a1a5c4+4a6c32−4a6c3c4+c32−c32c4+c42fy3+18h3k12a12a3+2a12a5+4a1a3c3+4a1a5c4+2a6c32+4a6c3c4−c32+c32c4+c3c42−c43fyyfy+112h3k12c3+c4c32−c32c4+c42fyyy*------------------(27)

$$k_3^2k_4^2 = k_1^4 + f_yk_1^4(c_3 + c_4)h + h^2k_1^4(2a_1a_3 + 2a_1a_5 + 2a_6c_3 - c_3^2 + 4c_3c_4 - c_4^2)f_y^2 + k_1^4h^2(c_3^2 + c_4^2)f_{yy} + 2h^3k_1^4(a_1a_3a_6 + a_1a_3c_3 + 2a_1a_3c_4 + 2a_1a_5c_3 + a_1a_5c_4 + 2a_6c_3^2 + a_6c_3c_4 + c_3^2c_4 + c_3c_4^2)f_y^3 + h^3k_1^5(a_1^2a_3 + a_1^2a_5 + 2a_1a_3c_3 + 2a_1a_5c_4 + a_6c_3^2 + 2a_6c_3c_4 + c_3^2 + 2c_3^2c_4 + 2c_3c_4^2 + c_4^3)f_{yy}f_y + \frac{1}{3}k_1^6h^3(c_3 + c_4)(c_3^2 - c_3c_4 + c_4^2)f_{yyy} ------(28)$$

International Journal of Scientific Research & Engineering Trends
Volume 6, Issue 5, Sept-Oct-2020, ISSN (Online): 2395-566X

$$\begin{aligned} k_3^3k_4^3 = k_1^6 &+ 3f_y k_1^6(c_3+c_4)h \\ &+ 3h^2k_1^6(a_1a_3 + a_1a_5 + a_6c_3 + c_3^2 \\ &+ 3c_3c_4 + c_4^2)f_y^2 + \frac{3}{2}k_1^7h^2(c_3^2+c_4^2)f_{yy} \\ &+ h^3k_1^6(3a_1a_3a_6 + 6a_1a_3c_3 \\ &+ 9a_1a_3c_4 + 9a_1a_5c_3 + 6a_1a_5c_4 \\ &+ 9a_5c_3 + 9a_6c_3^2 + 6a_6c_3c_4 + c_3^2 \\ &+ 9c_3^2c_4 + 9c_3c_4^2 + c_4^3)f_y^3 \\ &+ \frac{3}{2}h^3k_1^7(a_1^2a_3 + a_1^2a_5 + 2a_1a_3c_3 \\ &+ 2a_1a_5c_4 + a_6c_3^2 + 2a_6c_3c_4 + 2c_3^2 \\ &+ 3c_3^2c_4 + 2c_4^3)f_{yy}f_y \\ &+ \frac{1}{2}k_1^8h^3(c_3+c_4)(c_3^2 - c_3c_4 + c_4^2)f_{yyy} \\ &- - - - - (29) \end{aligned}$$

Substituting (27) to (29) into (25) and collecting like-terms we obtain;

$$\begin{aligned} \sqrt{k_3k_4} = 1 &+ \frac{1}{2}f_y(c_3+c_4)h \\ &+ \frac{1}{8}f_y^2(4a_1a_3 + 4a_1a_5 + 4a_6c_3 - c_3^2 \\ &+ 2c_3c_4 - c_4^2)h^2 + \frac{1}{4}f_{yy}k_1(c_3^2+c_4^2)h^2 \\ &+ \frac{1}{16}h^3(8a_1a_3a_6 - 4a_1a_3c_3 \\ &+ 4a_1a_3c_4 + 4a_1a_5c_3 - 4a_1a_5c_4 \\ &+ 4a_6c_3^2 - 4a_6c_3c_4 + c_3^2 - c_3^2c_4 - c_3c_4^2 \\ &+ c_4^3)f_y^3 \\ &+ \frac{1}{8}h^3k_1(2a_1^2a_3 + 2a_1^2a_5 + 4a_1a_3c_3 \\ &+ 4a_1a_5c_4 + 2a_6c_3^2 + 4a_6c_3c_4 - c_3^2 \\ &+ c_3^2c_4 + c_3c_4^2 - c_4^3)f_{yy}f_y \\ &+ \frac{1}{12}h^3k_1^2(c_3+c_4)(c_3^2 - c_3c_4 \\ &+ c_4^2)f_{yyy} - - - - - - - - (30) \end{aligned}$$

Substituting (21), (22), (23) and (30) into (10b) and simplifying we have;

$$\begin{aligned} y_{n+1} - y_n = h &+ \frac{1}{8}f_y(3a_1 + 3c_3 + c_4)h^2 \\ &+ \frac{1}{32}h^3(-a_1^2 + 12a_1a_3 + 4a_1a_5 \\ &+ 4a_6c_3 - c_3^2 + 2c_3c_4 - c_4^2)f_y^2 \\ &+ \frac{1}{16}h^3(3a_1^2 + 3c_3^2 + c_4^2)f_{yy} \\ &+ \frac{1}{64}h^4(a_1^3 + 8a_1a_3a_6 - 4a_1a_3c_3 \\ &+ 4a_1a_3c_4 + 4a_1a_5c_3 - 4a_1a_5c_4 \\ &+ 4a_6c_3^2 - 4a_6c_3c_4 + c_3^2 - c_3^2c_4 - c_3c_4^2 \\ &+ c_4^3)f_y^3 \\ &+ \frac{1}{32}h^4(-a_1^3 + 6a_1^2a_3 + 2a_1^2a_5 \\ &+ 12a_1a_3c_3 + 4a_1a_5c_4 + 2a_6c_3^2 \\ &+ 4a_6c_3c_4 - c_3^2 + c_3^2c_4 + c_3c_4^2 \\ &- c_4^3)f_{yy}f_y \\ &+ \frac{1}{48}h^4(3a_1^3 + 3c_3^3 + c_4^3)f_{yyy} - - - - \\ &- -(31) \end{aligned}$$

Finally, we compared equation (31) with Taylor series expansion of order four given by:

$$\begin{aligned} y_{n+1} - y_n = hk_1 + \frac{1}{2}h^2k_1f_y + \frac{1}{6}h^3\left(k_1f_y^2 + k_1^2f_{yy}\right) \\ + \frac{1}{24}h^4\left(k_1f_y^3 + 4k_1^2f_yf_{yy} + k_1^3f_{yyy}\right) \end{aligned}$$

to arrive at the following set of parametric equations;

$3a_1 + 3c_3 + c_4 = 4$------------------(32)

$-a_1^2 + 12a_1a_3 + 4a_1a_5 + 4a_6c_3 - c_3^2 + 2c_3c_4 - c_4^2 = \frac{16}{3}$------------------------(33)

$3a_1^2 + 3c_3^2 + c_4^2 = \frac{8}{3}$-----------------------(34)

$a_1^3 + 8a_1a_3a_6 - 4a_1a_3c_3 + 4a_1a_3c_4 + 4a_1a_5c_3 - 4a_1a_5c_4 + 4a_6c_3^2 - 4a_6c_3c_4 + c_3^2 - c_3^2c_4 - c_3c_4^2 + c_4^3 = \frac{8}{3}$-------------(35)

$-a_1^3 + 6a_1^2a_3 + 2a_1^2a_5 + 12a_1a_3c_3 + 4a_1a_5c_4 + 2a_6c_3^2 + 4a_6c_3c_4 - c_3^2 + c_3^2c_4 + c_3c_4^2 - c_4^3 = \frac{16}{3}$---(36)

$3a_1^3 + 3c_3^3 + c_4^3 = 2$ ---------------(37)

For ease computation and convenience, we set:

$$c_3 = \frac{2}{3}, \quad c_4 = 1, \quad a_1 = \frac{1}{3}$$

Solving the above equations with maple-18 package we obtain the values of a_2, a_3, a_4, a_5, and a_6 as follows.

$$a_2 = -\frac{509}{1500}, \quad a_3 = \frac{503}{500}, \quad a_4 = \frac{3931}{5000}, \quad a_5 = -\frac{721}{1000}, \quad a_6 = \frac{2337}{2500}$$

Substituting these values of $a_{i's}$ into equation (10b) with $k_{i's}$ defined in (6a) to (6d) we have the required formula given by;

$y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1k_2} + k_2 + k_3 + \sqrt{k_3k_4}\right)$---------(38)

With

$k_1 = f(y_n)$-------------(38a)

$k_2 = f\left(y_n + h\left(\frac{1}{3}k_1\right)\right)$---------------(38b)

$k_3 = f\left(y_n + h\left(-\frac{509}{1500}k_1 + \frac{503}{500}k_2\right)\right)$---------(38c)

$k_4 = f\left(y_n + h\left(\frac{3931}{5000}k_1 - \frac{721}{1000}k_2 + \frac{2337}{2500}k_3\right)\right)$------(38d)

The Butcher's table [1] will be:

$$
\begin{array}{c|ccc}
0 & 0 & & \\
\frac{1}{3} & \frac{1}{3} & & \\
\frac{2}{3} & -\frac{509}{1500} & \frac{503}{500} & \\
1 & \frac{3931}{5000} & -\frac{721}{1000} & \frac{2337}{2500} \\
\hline
\end{array}
$$

III. IMPLEMENTATION OF OCB4 METHOD

In this section, the efficiency and suitability of the OCB4 method is illustrated. The result are compared with other related methods in literature such as Kutta's method based on Arithmetic mean denoted as (KMAM), Runge-Kutta method based on Geometric mean written as (RKMGM) and the proposed method based on Geometric mean which is denoted as (OCB4).The numerical solutions to these initial values were generated by a well thought out MATLAB package.

Problem 1. $y' = \frac{1}{y}; \quad y(0) = 1, \quad 0 \le x \le 1$ with theoretical solution $y(x) = \sqrt{2x+1}, h = 0.1$[2]

Problem2. $y' = y^2; \quad y(0) = 1, \quad 0 \le x \le 1$ with theoretical solution $y(x) = -\frac{1}{x-1}, h = 0.1$ [5]

Problem 3. $y' = 3x + y; \quad y(0) = 1, 0 \le x \le 1$ with theoretical solution $y(x) = -3x - 3 + 4e^x, \; h = 0.1$

The numerical results are presented below;

Table 1 Numerical Result for problem 1.

		KMAM		RKMGM		OCB4	
XN	TSOL	YN	ERROR	YN	ERROR	YN	ERROR
0.1	1.0954	1.0954	1.1030E-07	1.0954	7.5511E-09	1.0954	1.7500E-08
0.2	1.1832	1.1832	1.5245E-07	1.1832	1.0727E-08	1.1832	2.5603E-08
0.3	1.2649	1.2649	1.6843E-07	1.2649	1.2064E-08	1.2649	2.9415E-08
0.4	1.3416	1.3416	1.7324E-07	1.3416	1.2560E-08	1.3416	3.1125E-08
0.5	1.4142	1.4142	1.7299E-07	1.4142	1.2649E-08	1.4142	3.1752E-08
0.6	1.4832	1.4832	1.7039E-07	1.4832	1.2535E-08	1.4832	3.1800E-08
0.7	1.5492	1.5492	1.6672E-07	1.5492	1.2321E-08	1.5492	3.1533E-08
0.8	1.6125	1.6125	1.6262E-07	1.6125	1.2060E-08	1.6125	3.1095E-08
0.9	1.6733	1.6733	1.5842E-07	1.6733	1.1779E-08	1.6733	3.0567E-08
1	1.7321	1.7321	1.5429E-07	1.7321	1.1496E-08	1.7321	2.9998E-08

From the numerical result for problem 1, it is very clear that OCB4 method shows superiority over the Kutta's method based on Arithmetic Mean. It can be shown that the OCB4 method competes favorably with other existing methods in terms of accuracy.

Table 2 Numerical Result for problem 2.

		KMAM		RKMGM		OCB4	
XN	TSOL	YN	ERROR	YN	ERROR	YN	ERROR
0.1	1.1111	1.1111	5.5094E-07	1.1111	2.9316E-06	1.1111	2.1261E-06
0.2	1.25	1.25	1.8000E-06	1.25	9.3540E-06	1.25	6.7984E-06
0.3	1.4286	1.4286	4.7602E-06	1.4285	2.3995E-05	1.4286	1.7494E-05
0.4	1.6667	1.6667	1.2369E-05	1.6666	5.9918E-05	1.6666	4.3886E-05
0.5	2	2	3.4533E-05	1.9998	1.5869E-04	1.9999	1.1705E-04
0.6	2.5	2.4999	1.1248E-04	2.4995	4.8109E-04	2.4996	3.5883E-04
0.7	3.3333	3.3329	4.8023E-04	3.3315	1.8571E-03	3.3319	1.4114E-03
0.8	5	4.9966	3.3521E-03	4.9888	1.1159E-02	4.9912	8.7793E-03
0.9	10	9.9312	6.8831E-02	9.824	1.7596E-01	9.8511	1.4893E-01
1	9.0072E+15	85.2571	9.0072E+15	63.8064	9.0072E+15	65.9959	9.0072E+15

From the numerical results for problem 2, above, it can be shown that the OCB4 method competes favorably well with other existing methods in terms of accuracy.

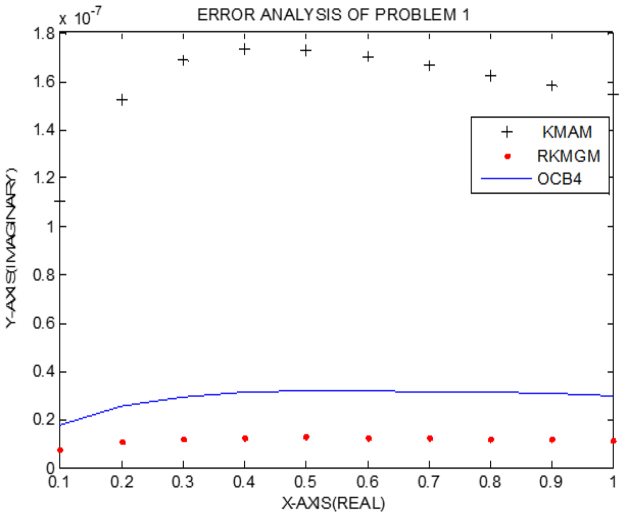
Table 3 Numerical Result for problem 3.

		KMAM		RKMGM		OCB4	
XN	TSOL	YN	ERROR	YN	ERROR	YN	ERROR
0.1	1.1207	1.1214	7.3716E-04	1.1205	2.1151E-04	1.1213	6.0927E-04
0.2	1.2856	1.2874	1.8336E-03	1.2852	4.0459E-04	1.2872	1.5933E-03
0.3	1.4994	1.5028	3.3570E-03	1.4988	5.9279E-04	1.5024	3.0107E-03
0.4	1.7673	1.7727	5.3853E-03	1.7665	7.8375E-04	1.7722	4.9346E-03
0.5	2.0949	2.1029	8.0079E-03	2.0939	9.8256E-04	2.1023	7.4516E-03
0.6	2.4885	2.4998	1.1328E-02	2.4873	1.1931E-03	2.4991	1.0662E-02
0.7	2.955	2.9705	1.5463E-02	2.9536	1.4188E-03	2.9697	1.4683E-02
0.8	3.5022	3.5227	2.0548E-02	3.5005	1.6626E-03	3.5218	1.9648E-02
0.9	4.1384	4.1652	2.6738E-02	4.1365	1.9276E-03	4.1641	2.5709E-02
1	4.8731	4.9073	3.4209E-02	4.8709	2.2169E-03	4.9062	3.3043E-02

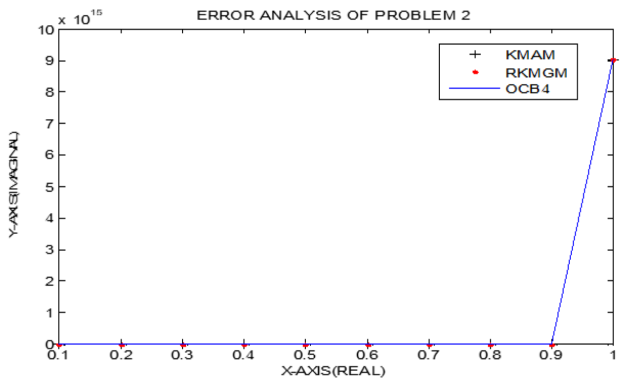
Finally, from the above numerical result for problem 3, OCB4 method clearly shows superiority over the Kutta's method based on Arithmetic mean, it can be shown that the OCB4 method competes favorably with other existing methods in terms of accuracy

IV. ERROR ANALYSIS OF OCB4

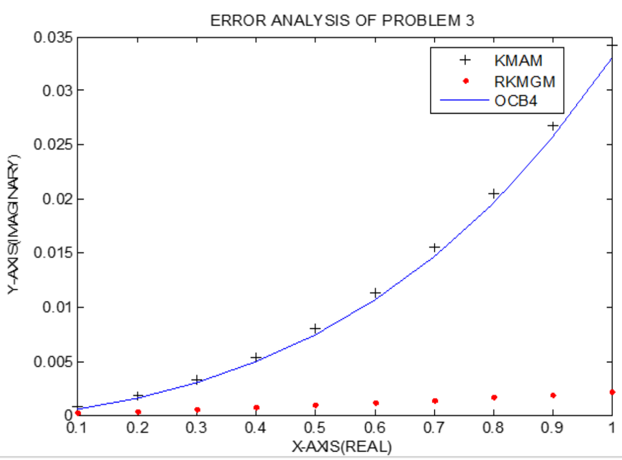
In this section, we present the error trajectories of the OCB4 with other existing methods. Below are the plots that show the error analysis for problem 1, problem 2 and problem 3.



PLOT 1: Shows The Error Analysis Of Problem 1
From Plot 1 above, it is very clear that the propose OCB4 method performs better in terms of accuracy than Kutta's method based on the Arithmetic Mean.



PLOT 2: Shows The Error Analysis Of Problem 2
From Plot 2, above, it can be shown that the OCB4 method competes favorably with other existing methods in terms of accuracy.



PLOT 3: Shows The Error Analysis Of Problem 3
Finally, from plot 3, OCB4 clear shows superiority over the Kutta's method based on Arithmetic mean, it can be shown that the OCB4 competes favorably with other existing methods in terms of accuracy.

V. SUMARRY AND CONCLUSION

In this paper, the derivation of a new fourth order Kutta's formula based on Geometric Mean has been successfully carried out. Several practically applicable problems have been considered to test the suitability, adoptability and accuracy of the OCB4 method. To achieve this, three test problems were consider and the results indicated that the OCB4 method is of high degree of accuracy in comparison with other existing methods, this shows that the OCB4 method can be use to solve real life problem, that can possibly be reduced to first order ordinary differential equations. Furthermore, this method was seen to be effective in solving linear and non-stiff problems, as can be seen from the given problems above. We will consider in the next publication the consistency and convergence level of the OCB4 method as well as the stability.

REFERENCE

[1] Ononogbo, .C.B and Aziegbemi,. B. O. "On the Derivation and Implementation of a Fifth-Stage Fourth-Order Runge–Kutta Formula for Solving Initial Value Problems in Ordinary Differential Equations." IOSR Journal of Mathematics (IOSR-JM), 16(4), (2020): pp. 29-39.
[2] RiniYanti, M Imran, Syamsudhuha"A Third Runge Kutta method based on a linear combination of arithmetic mean, harmonic mean and geometric mean" .Journal of Applied and Computation 2014;3(5):231-234
[3] Bazuaye Frank, Etin-Osa, A New 4th Order Hybrid Runge-Kutta Method for Solving Initial Value Problems (IVP), Pure and Applied Mathematics Journal. Vol.7, No.6, 2018, pp 78-87. Doi: 10:116481j.20180706.11
[4] AashikpelokhaiU .S. U and Agbeboh, G. U." On the Analysis of a Cubic Root Mean 4th order Runge-Kutta Formula", African Journal of Science (AJS) 2005, 6(1),pp 1310-1318.
[5] Agbeboh, G. U. ,Aashikpelokhai U .S. U and I. Aigbedion(2007):Implementation of a new 4th order Kunge- Kutta formula for solving initial value problems (I.V.Ps). International journal of physical sciences, 2(4) pp.089-098.
[6] Agbeboh, And Esekhaigbe On the Component Analysis and transformation of an explicit fifth stage forth order Runge-Kutta methods. International Journal of mathematics Research, 2 January, 2016.vol4,no.2 pp.76-100

[7] Agbeboh, And Omonkaro. On the solution of singular initial value problems in ordinary Differential equations using a new third order inverse Runge-Kutta method. International Journal of Physical Sciences. 2009 vol. 5(4), pp. 299-307.

[8] AshiroboSenaponWusu, Moses Adebowale Akanbi, BakreOmolareFalinah. “On the Derivation and Implementation of a four stage Harmonic Explicit Runge-Kutta method. Applied Mathematics, 2015,6,694-699.

[9] Osana Yusuf Ababreh and Rokiah Rozita “New third order Runge-Kutta Base on Contraharmonic mean for stiff problems. Applied Mathematics Science Vol 3, 2009, No 8, 365-376.

[10] Refiqul.Islam and Jalal, Uddin. “Reformation on Modified Runge-Kutta third order method for solving Initial Value Problem. International Journal of Engineering Research and Technology.(IJERT). Vol 4, May, 2015.

[11] Lambert.J.D. Computational Methods in Ordinary Differential Equations. 1973 pp 114-149, John Wiley and Sons, London New York Sydney Toronto

[12] Agbeboh G.U. (2013): “On the stability analysis of a geometric 4th order Runge-Kutta formula” Journal Mathematical theory and modeling Vol. 3, pp. 90-105,2013.