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# Inequality Concerning the Polar Derivative of a Polynomial

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***Abstract* - For the present paper, we will give a correct proof of $L_p$-inequality concerning the polar derivative of a polynomial with restricted zeros. We will also extend Zygmund's inequality to the polar derivative of a polynomial.**

***Keywords*- Polar derivative, $L_p$-norm inequalities, Zygmund's inequality.**

## I. INTRODUCTION

Suppose H (t) be a polynomial of degree m and suppose $H'$ be its derivative. According to the famous result known as Bernstein's inequality (see [7] or [2])

$$\max_{|t|=1}|H'(t)| \le m \max_{|t|=1} \qquad (1)$$

Inequality (1) is sharp and equality (1) holds good for $H(t) = bt^m, b \ne 0.$ Inequality (1) extended to $L_p$ -norm by zygmund [1] which shows if H(t) is a polynomial of degree m, $h \ge 1$,

$$\left\{\int_0^{2\pi}\left|H'\left(e^{i\theta}\right)\right|^h d\theta\right\}^{1/h} \le m\left\{\int_0^{2\pi}\left|H\left(e^{i\theta}\right)\right|^h d\theta\right\}^{1/h} \qquad (2).$$

We get the result which is sound and the equality in (2) holds for $H(t) = bt^m, b \ne 0.$ Let us suppose $h \to \infty$ in (2), we get inequality (1).

Suppose $P_\beta H(t)$ denotes the polar differentiation of polynomial H(z) with respect to a real or complex number $\beta$ . Then

$$P_\beta H(t) = m\,H(t) + (\beta - t)\,H'(t).$$

The polynomial $P_\beta H(t)$ is of degree at most $m-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{P_\beta H(z)}{\beta} = H'(t)$$

Uniformly on compact subsets of C.

As an extension of (1) to the polar derivative, Aziz and Shah (Theorem 4 with $k=1$, [8]) have shown that if H (z) is a polynomial of degree m, then for every complex number $\beta$ with $|\beta| \ge 1$,

$$|P_\beta H(t)| \le m|\beta| \max_{|t|=1}|H(t)| \quad for\ |t|=1 \qquad (3)$$

Inequality (3) becomes equality for $H(t) = bt^m, b \ne 0.$ If we divide the both side of the (3) by $\beta$ and suppose $\beta \to \infty,$ we get the inequality (1).

It is natural to seek $L_p$-norm analog of inequality (3). In view of the $L_p$ -norm extension (2) of inequality (1), one would expect that if H(t) is a polynomial of degree m, then

$$\left[\int_0^{2\pi}\left|P_\beta H(e^{i\theta})\right|^h d\theta\right]^{\frac{1}{h}} \le m|\beta|\left[\int_0^{2\pi}\left|H(e^{i\theta})\right|^h\right]^{\frac{1}{h}} \qquad (4)$$

Will be $L_p$ -norm extension of (3) analogous to (2). But unfortunately inequality (4) is not, in general, true for every real or complex number $\beta$ . To see this, we will take in particular $h=2, H(t)=(1-it)^m$ and $\beta = i\gamma$ where $\gamma$ is any positive real number such that

$$1 \le \gamma < \frac{n+\sqrt{2m(2m-1)}}{3m-2} \qquad (5)$$

Now

$$P_\beta H(t) = m(1-iz)^m - m\,i(\beta - t)(1-it)^{m-1}$$
$$= m(1-it)^{m-1}(1-i\beta)$$

So that

$$\int_0^{2\pi}\left|P_\beta H(e^{i\theta})\right|^h d\theta = m^2|1-i\beta|^2\int_0^{2\pi}\left|1-ie^{i\theta}\right|^{2(m-1)} d\theta$$
$$= m^2|1-i\beta|^2\int_0^{2\pi}\left|(1-ie^{i\theta})^{m-1}\right|^2 d\theta$$

$$= m^2|1-i\beta|^2\int_0^{2\pi}\left|\begin{array}{l}\binom{m-1}{0}-\binom{m-1}{1}(ie^{i\theta})\\ +\binom{m-1}{2}(ie^{i\theta})^2\\ -\ldots\\ +(-1)^{m-1}\binom{m-1}{m-1}(ie^{i\theta})^{m-1}\end{array}\right|^2 d\theta$$

$$= 2\pi m^2|1-i\beta|^2\left(\begin{array}{l}\binom{m-1}{0}^2\\ +\binom{m-1}{1}^2\\ +\binom{m-1}{2}^2\\ +\ldots+\binom{m-1}{m-1}^2\end{array}\right)$$

$$= 2\pi m^2|1-i\beta|^2\binom{2(m-1)}{m-1} \qquad (6)$$

also

$$m^2|\beta|^2\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta = m^2|\beta|^2\int_0^{2\pi}\left|1-e^{i\theta}\right|^{2m} d\theta$$

$$= m^2|\beta|^2\int_0^{2\pi}\left|(1-e^{i\theta})^m\right|^2 d\theta$$

$$= m^2|\beta|^2\int_0^{2\pi}\binom{m}{0}-\binom{m}{1}(ie^{i\theta})$$

$$+\binom{m}{2}(ie^{i\theta})^2-\ldots+$$

$$(-1)^m\binom{m}{m}(ie^{i\theta})^{2m}d\theta \qquad (7)$$

Using (6) and (7) in (4),we get

$$= 2\pi m^2|\beta|^2\left(\binom{m}{0}^2+\binom{m}{1}^2+\ldots+\binom{m}{m}^2\right)$$

This implies

$$m|1-i\beta|^2 \le 2(2m-1)|\beta|^2 \qquad (8)$$

Setting β=iγ in (8), we get

$$m(1+\gamma)^2 \le 2(2m-1)\gamma^2$$

This inequality can be written as

$$\left(\gamma-\frac{m+\sqrt{2m(2m-1)}}{3m-2}\right)\left(\gamma-\frac{m-\sqrt{2m(2m-1)}}{3m-2}\right)\ge 0 \qquad (9)$$

Sinc $\gamma \ge 1$, we have

$$\left(\gamma-\frac{m-\sqrt{2m(2m-1)}}{3m-2}\right)\ge 1-\frac{m-\sqrt{2m(2m-1)}}{3m-2}$$

$$=\frac{2(m-1)+\sqrt{2m(2m-1)}}{3m-2}>0$$

and hence from (9), it follows that

$$\gamma-\frac{m-\sqrt{2m(2m-1)}}{3m-2}\ge 0.$$

This gives

$$\gamma \ge \frac{m-\sqrt{2m(2m-1)}}{3m-2}$$

It gets contradicts (5). Hence inequality (4) is not, in general. True for all polynomials H(t) of degree $m \ge 1$. However, we have been able to prove the following generalization of (2) to the polar derivatives.

**Theorem 1.** If H(t) is a polynomial of degree m, then for every complex number $\beta$ and $h \ge 1$,

$$\left[\int_0^{2\pi}\left|P_\beta H(e^{i\theta})^h d\theta\right|\right]^{\frac{1}{h}} \le m(|\beta|+1)\left[\int_0^{2\pi}\left|H(e^{i\theta})^h d\theta\right|\right]^{\frac{1}{h}} \qquad (10)$$

**Remark.** If we divide both sides of (10) by $|\beta|$ and make $|\beta \to \infty|$, we get inequality (2) due to Zygmund [1].

For polynomials H(t) which does not vanish in the unit disk, the right hand side of (2) can be improved. In fact, in this direction, it was shown by De-Bruijn [4] that if H(t) does not vanish in $|z|<1$,

$$\left[\int_0^{2\pi}\left|H'(e^{i\theta})^h d\theta\right|\right]^{\frac{1}{h}} \le mC_p\left[\int_0^{2\pi}\left|H(e^{i\theta})^h d\theta\right|\right]^{\frac{1}{h}} \qquad (11)$$

Where $C_p=\left[\frac{1}{2}\int_0^{2\pi}\left|1+e^{i\gamma}\right|^h d\gamma\right]^{-\frac{1}{h}}$ (12)

Inequality (11) is best possible with equality for $H(t)=bt^m+c, |b|=|c|$.If we let $h\to\infty$ in (11), it follows that if $H(t)\ne 0 \; for |t|<1, then$

$$\max_{|t|=1}|H'(t)| \le \frac{n}{2}\max_{|t|=1}|H(t)| \qquad (13)$$

Inequality (13) was conjectured by Erdos and later verified by Lax[3]. Aziz [6] extended (13) to the polar derivative of a polynomial and proved that if H(t) is a polynomials of degree m which does not vanish in $|t|<1$, then for every complex number $\beta$ with $|\beta|\geq 1$,

$$\max_{|t|=1}|P_{\beta}H(t)|\leq\frac{n}{2}(|\beta|+1)\max_{|t|=1}|H(t)| \qquad (14)$$

This estimate (14) is best possible with equality for $H(t)=t^m+1$ If we divide both sides of (14) by $|\beta|$ and make $|\beta|\rightarrow\infty$, we get inequality (13) due to Lax [3]. While seeking the desired extension to the polar derivatives, recently Govil et al [10] have made an incomplete attempt by claiming to have proved the following generalization of (11) and (14).

**Theorem2**. If H(t) is a polynomials of degree m which does not vanish $|t|<1$, in then for every complex number $\beta$ with $|\beta|\geq 1$,

$$\left[\int_0^{2\pi}|P_{\beta}H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}}\leq m(|\beta|+1)C_p\left[\int_0^{2\pi}|H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}} \quad (15)$$

Where $C_p$ is defined by (12). Unfortunately the proof of this theorem, which is the main result (theorem 1.1 of [10]) given by govil, Nyuydinkong and Tameru is not correct, because the claim made by the authors on page 624 in lines 12 to 16 by using Lema 2.3 is incorrect. The reason being that their polynomial.

$$P_{\beta}H_m(t)+e^{i\gamma}\{m\overline{\beta}tH_m(t)+(1-\overline{\beta}t)tH_m'(t)\},t=e^{i\theta}$$

In general does not take the form

$$\sum_{k=0}^{m}l_kb_kt^k,t=e^{i\theta}$$

where $H_n(t)=\sum_{k=0}^{m}b_kt^k$ and the complex number $l_k$ defined by them on page 624, line 10, by

$$L(H_m(e^{i\theta}))=\left[\Lambda H_m(e^{i\theta})\right]_{\theta=0}=\sum_{k=0}^{m}l_ka_k$$

Along with the equation (24) of [4]. It is worthwhile to note here that if we take

$$L(H_m(e^{i\theta}))=\{mH_m(e^{i\theta})+(\beta-e^{i\theta})H_m'(e^{i\theta})\}_{\theta=0}$$

and use the same argument as user by Govil et al (page 624, line 10 of [4]), then in view of the inequality

$$|P_{\beta}H(t)|\leq m|\beta|\max_{|t|=1}|H(t)| \quad for|t|=1$$

(see Theorem 4 with k=1 of [8], the above bounded functional has norm $N\leq m|\beta|$

Therefore, if we use lemma 2.3 of [4] which is due to Rahman (lemma 3 of [5]), it would follow that

$$\left[\int_0^{2\pi}|P_{\beta}H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}}\leq m|\beta|\left[\int_0^{2\pi}|H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}}$$

For every h $\geq$1 and $|\beta|\geq$1, and which is not true in general as shown above.

Here we shall also present a correct proof of theorem 2, which shall validate Theorems 1.2 and 1.3 of Govil el al [10] as well. Finally we shall also present a short proof of Theorem 1.3 of [10]. That is, we prove the following.

**Theorem 3.** If H(t) is a self inverse polynomial of degree m, then for every complex number $\beta$ and $h\geq 1$,

$$\left[\int_0^{2\pi}|P_{\beta}H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}}\leq m(|\beta|+1)C_p\left[\int_0^{2\pi}|H(e^{i\theta})^h|d\theta\right]^{\frac{1}{h}} \quad (16)$$

Where $C_p$ is the same as in Theorem2.

## II. LEMMAS

For the proofs of the theorem, we need the following lemmas.

**Lemma 1.** If H(t) is a polynomial of degree m which does not vanish $|t|<1,\ and\ \ Q(t)=t^m\overline{H\left(\frac{1}{t}\right)}$, then for every complex number $\beta$ with $|\beta|\geq 1$,

$$|P_{\beta}H(t)|\leq|P_{\beta}Q^*(t)| \quad for\ \ |t|\geq 1.$$

Lemma 1 is due to Aziz (p.190 of [6]

**Lemma 2.** If H(t) is a polynomial of degree m and $Q^*(t)=t^n\overline{H\left(\frac{1}{\overline{t}}\right)}$, then for every $h\geq 0$ and $\beta$ real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|Q^*(e^{i\theta})+e^{i\beta}H'(e^{i\theta})\right|^h d\theta d\beta\leq 2\pi m^p\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta$$

Lemma2 is due to Aziz [11] (see also [5]. We also need the following lemma.

**Leema3.** If H(t) is a polynomial of degree m, $H(0)\neq 0$ and $Q^*(t)=t^m\overline{H\left(\frac{1}{\overline{t}}\right)}$, then for every complex number $\beta,h\geq 1$ and $\beta$ real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|P_{\beta}Q^*(e^{i\theta})+e^{i\gamma}P_{\beta}H(e^{i\theta})\right|^h d\theta d\beta$$
$$\leq 2\pi m^h(|\beta|+1)^h\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta$$

**Proof of lemma 3**. We have by Minkowski's inequality for every $h\geq 1$ and $\beta$ real,

$$\left[\int_0^{2\pi}\int_0^{2\pi}\left|P_\beta Q^*(e^{i\theta})+e^{i\gamma}P_\beta H(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$=\left[\int_0^{2\pi}\int_0^{2\pi}\left|\begin{matrix} mQ^*(e^{i\theta})+(\beta-e^{i\gamma})Q^{*'}(e^{i\theta}) \\ \\ +e^{i\gamma}(mH(e^{i\theta})+(\beta-e^{i\theta})H'(e^{i\theta})\end{matrix}\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$=\left[\int_0^{2\pi}\int_0^{2\pi}\left|\begin{matrix} mQ^*(e^{i\theta})-e^{i\gamma}Q^{*'}(e^{i\theta})+e^{i\gamma}(mH(e^{i\theta}) \\ -e^{i\theta}H'(e^{i\theta})+\beta(Q^{*'}(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\end{matrix}\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$\le\left[\int_0^{2\pi}\int_0^{2\pi}\left|\begin{matrix} mQ^{*'}(e^{i\theta}) \\ -e^{i\gamma}Q^{*'}(e^{i\theta}) \\ +e^{i\gamma}(mH(e^{i\theta}) \\ -e^{i\theta}H'(e^{i\theta})\end{matrix}\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$+|\beta|\left[\int_0^{2\pi}\int_0^{2\pi}\left|Q^{*'}(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}} \quad (17)$$

Since $Q^{*'}(t)=t^m\overline{H\left(\frac{1}{\bar t}\right)}$, we have $H(t)=t^m\overline{Q^*\left(\frac{1}{\bar t}\right)}$ and it can be easily verified that for $0\le\theta<2\pi$

$$mH(e^{i\theta})-e^{i\theta}H'(e^{i\theta})=e^{i(m-1)\theta}\overline{Q^{*'}(e^{i\theta})} \quad (18)$$

and $mQ^*(e^{i\theta})-e^{i\theta}Q^{*'}(e^{i\theta})=e^{i(m-1)\theta}\overline{H'(e^{i\theta})}$ (19)

Using (18) and (19) in (17), we obtain

$$\left[\int_0^{2\pi}\int_0^{2\pi}\left|P_\beta Q^*(e^{i\theta})+e^{i\beta}P_\beta H(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$\le\left[\int_0^{2\pi}\int_0^{2\pi}\left|e^{i(m-1)\theta}\overline{H'(e^{i\theta})}+e^{i\gamma}e^{i(m-1)\theta}\overline{Q^{*'}(e^{i\theta})}\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$+|\beta|\left[\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$=\left[\int_0^{2\pi}\int_0^{2\pi}\left|Q^{*'}(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$+|\beta|\left[\int_0^{2\pi}\int_0^{2\pi}\left|Q^{*'}(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

$$=(|\beta|+1)\left[\int_0^{2\pi}\int_0^{2\pi}\left|Q^{*'}(e^{i\theta})+e^{i\gamma}H'(e^{i\theta})\right|^h d\theta d\gamma\right]^{\frac{1}{h}}$$

This gives with the help of lemma2,

$$\int_0^{2\pi}\int_0^{2\pi}P_\beta Q(e^{i\theta})+e^{i\theta}P_\beta H(e^{i\theta})^h d\theta d\gamma$$

$$\le 2\pi m^p(\beta+1)^h\int_0^{2\pi}\left|H(e^{i\theta})\right|h d\theta$$

This completes the proof of Lemma3.

## III. PROOF OF THE THEOREMS

**Proof of Theorem 1**. By Lemma3, we have for every complex number $\beta, h\ge 1 and \beta$ real

$$\int_0^{2\pi}\int_0^{2\pi}P_\beta Q^*(e^{i\theta})+e^{i\theta}P_\beta H(e^{i\theta})^h d\theta d\gamma$$

$$\le 2\pi m^p(|\alpha|+1)^h\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta \quad (20)$$

Using in (20) the fact that for any $h\ge 0$,

$$\int_0^{2\pi}\left|b+ce^{i\beta}\right|^h d\beta\ge 2\pi\max\left(|b|^h,|c|^h\right)$$

(See inequality (19) of [5]), we obtain

$$\left[\int_0^{2\pi}\left|P_\beta H(e^{i\theta})\right|^h\right]^{\frac{1}{h}}\le m(|\beta|+1)\left[\int_0^{2\pi}\left|H(e^{i\theta})^h\right|d\theta\right]^{\frac{1}{h}}$$

This completes the proof of theorem 1.

**Proof of Theorem 2.** Since H(t) is a polynomial of degree m which does not vanish in $|h|<1$, by Lemma 1 we have for every complex number $\beta$ with $|\beta|\ge 1$

$$\left|P_\beta H(t)\right|\le\left|P_\beta Q^*(t)\right| \quad for \quad |t|=1 \quad (21)$$

Where $Q^*(t)=tm\overline{Q^*\left(\frac{1}{\bar t}\right)}$. Also by Lemma 3 for every complex number $\beta, h\ge 1$ and $\gamma$ real,

$$\int_0^{2\pi}\left[\int_0^{2\pi}\left|P_\beta Q^*(e^{i\theta})+e^{i\gamma}P_\beta(e^{i\theta})\right|^h d\beta\right]d\theta$$

$$\le 2\pi m^p(|\beta|+1)^h\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta \quad (22)$$

Now for every real β and r≥1, we have

$\left|r+e^{i\beta}\right|\ge\left|1+e^{i\beta}\right|$, Which implies

$$\int_0^{2\pi}\left|r+e^{i\gamma}\right|^h d\beta \geq \int_0^{2\pi}\left|1+e^{i\beta}\right|^h d\gamma \;\; h\geq 0$$

If $P_\beta H(e^{i\theta})\neq 0$ ,we take $r=\dfrac{\left|P_\beta Q^*(e^{i\theta})\right|}{P_\beta H(e^{i\theta})}$,

and by (21) $r\geq 1$

$$\int_0^{2\pi}\left|P_\beta Q^*(e^{i\theta})+e^{i\beta}P_\beta H(e^{i\theta})\right|^h d\beta$$

$$=P_\beta Q^*(e^{i\theta})\int_0^{2\pi}\left|\frac{P_\beta Q^*(e^{i\theta})}{P_\beta Q^*(e^{i\theta})}+e^{i\gamma}\right|d\gamma$$

$$=P_\beta Q^*(e^{i\theta})\int_0^{2\pi}\left|\left|\frac{P_\beta Q^*(e^{i\theta})}{P_\beta Q^*(e^{i\theta})}\right|+e^{i\gamma}\right|d\gamma$$

$$\geq\left|P_\beta H(e^{i\theta})\right|^h\int_0^{2\pi}\left|1+e^{i\gamma}\right|^h d\gamma$$

$For\geq P_\beta H(e^{i\theta})=0$, this inequality is trivially true. Using this in (22), we conclude that for every complex number $\beta$ with $|\beta|\geq 1$ and $h\geq 1$,

$$\int_0^{2\pi}\left|1+e^{i\gamma}\right|^h d\gamma\int_0^{2\pi}\left|P_\beta H(e^{i\theta})\right|^h d\theta$$

$$\leq 2\pi m^p(|\beta|+1)\int_0^{2\pi}\left|H(e^{i\theta})\right|^h d\theta,$$

Which immediately leads to (15) and this completes the proof of theorem 2.

**Proof of Theorem3.** Since H(t) is a self inversive polynomials of degree m, we have $H(t)=Q^*(t)$ where $Q^*(t)=t^m\overline{H\frac{1}{t}}$ Therefore, for every complex number $\beta$, $\left|P_\beta H(t)\right|=\left|P_\beta Q^*(t)\right|$ for all $t\in C$,

So that

$$\left|\frac{P_\beta Q^*(e^{i\theta})}{P_\beta H(e^{i\theta})}\right|=1$$

Using this in (22) and proceeding similarly as in the proof of theorem 2, we get (16) and this proves Theorem3.

## REFERENCES

[1] (1932) Zygmund A, A remark on conjugate series, Proc. London Math. Soc. 34 (1932) 392–400
[2] (1941) Schaeffer A C, Inequalities of A. Markoff and S.N. Bernstein for polynomials and related functions, Bull. Am. Math. Soc. 47 (1941) 565–579
[3] (1944) Lax P D, Proof of a conjecture of P. Erdo¨s on the derivative of a polynomial, Bull. Am. Math. Soc. 50 (1944) 509–513
[4] (1947) De-Bruijn N G, Inequalities concerning polynomials in the complex domain, Nederl. Akad. Wetensch. Proc. 50 (1947) 1265–1272; Indag. Math. 9 (1947) 591–598
[5] (1969) Rahman Q I, Functions of exponentials type, Trans. Am. Math. Soc. 135 (1969) 295–309
[6] (1988) Aziz A, Inequalities for the polar derivative of a polynomial, J. Approx. Theory 55 (1988) 183–193
[7] (1994) Milovanovic´ G V, Mitrinovic´ D S and Rassias Th, Topics in polynomials: Extremal properties, inequalities, zeros (Singapore: World Scientific) (1994)
[8] (1998) Aziz A and Shah W M, Inequalities for the polar derivative of a polynomial, Indian J. Pure Appl. Math. 29 (1998) 163–173
[9] (1998) Rather N A, Extremal properties and location of the zeros of polynomials, Ph.D. thesis, submitted to the University of Kashmir (1998)
[10] (2001) Govil N K, Nyuydinkong G and Tameru B, Some Lp inequalities for the polar derivative of a polynomial, J. Math. Anal. Appl. 254 (2001) 618–626
[11] (2004) Aziz A and Rather N A, Some Zygmund type Lq inequalities for polynomials, J. Math.Anal. Appl. 289 (2004) 14–29