Factorization in Integral Domains

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Abstract: The M-evaluated spaces $R = M R_m$, which are nearly Schreier are classified under the suspicion that the fundamental conclusion $R$ of $\mathcal{R}$ is a root augmentation of $\mathcal{R}$, where $M$ is a without torsion, commutative, cancellative monoid. For the situation that $\mathcal{D}[M]$ is a commutative monoid area it is demonstrated that if $M$ funnel shaped and $\mathcal{D}[M] \subseteq \mathcal{D}[M]$ is a root augmentation, at that point $\mathcal{D}[M]$ is nearly Schreier if and just if $M$ and $\square D$ are nearly Schreier. On the off chance that $R = \mathcal{E}[n x]$ is a request in a quadratic augmentation field $\langle d \rangle$ of $\mathcal{S}$, it is demonstrated that the conditions; $\mathcal{R}[X]$ is IDPF; $\mathcal{R}[X]$ is inside factorial; $\mathcal{R}[X]$ is nearly Schreier; $\mathcal{R}[X] \subseteq \mathcal{R}[X]$ is a root expansion; and each prime divisor of $n$ likewise partitions the discriminant of the augmentation $K/\mathcal{S}$ are equal conditions.

Keywords: Almost Schreier; Atomic domain; Inside factorial; Pre-Schreier domain.

I. INTRODUCTION

A vital space $\mathcal{R}$ is said to be nuclear if each nonzero non-unit of $\mathcal{R}$ has a factorization into a result of a finite number of final components (iot as). A fundamental area $\mathcal{R}$ is said to be an IDPF space if each nonzero component of $\mathcal{R}$ has at most a finite number of non-relate unchanged divisors. These were first contemplated by Grams and Warner in their 1975 article [10] and have gotten very piece of consideration from that point forward. In [13], Malcomson and Okoh talk about a thought of factorization that is a more grounded condition on an indispensable space than the IDF property, however weaker than the special factorization property. Given a nonzero, non-unit component an every a necessary space $\mathcal{R}$, let $\mathcal{D}_{n}(a)$ mean the arrangement of non-relate final divisors of an. The area $\mathcal{R}$ is said to be an IDPF space (for unchanged divisors of forces finite) if for each nonzero, non-unit $a \in \mathcal{R}$, the association $\mathcal{P}_n \mathcal{D}_{n}(a)$ is finite, up to partners. Krull areas are IDPF [13, Corollary 3.3]. Recall that a ring augmentation $\mathcal{R} \subseteq \mathcal{T}$ is a root expansion if for every $t \in \mathcal{T}$, there exists $n \in \mathcal{N}$ with the end goal that $t_n \in \mathcal{R}$. By [8, Theorem 2.8], a Noetherian area $\mathcal{R}$ with vital conclusion $\mathcal{R}$ and nonzero conductor $(\mathcal{R})$ is IDPF if and just if the remainder of unit bunches $\mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{R})$ is a finite gathering and $\mathcal{R}$ is a root augmentation.

In [5], P. M. Cohn defined a nonzero component $p$ of an essential area $\mathcal{R}$ to be primal if at whatever point $p \mid x y$, $x, y \in \mathcal{R}$, there exist $p_1, p_2 \in \mathcal{R}$ to such an extent that $p_1 p_2$ with $p_1(x)$ and $p_2(y)$, and defined a space to be Schreier in the event that it is vitally shut and each nonzero component of $\mathcal{R}$ is primal. In [16], M. Zafrullah presented the helpful wording of pre-Schreier for an area $\mathcal{R}$ in which each nonzero component of $\mathcal{R}$ is prim

In their 2010 paper [6], Dumitrescu and Khalid define a nonzero element $p$ of an integral domain $\mathcal{R}$ to be almost primal if whenever $| p_a | x, y \in \mathcal{R}$, there exists an integer $k \geq 1$ and $p_1, p_2 \in \mathcal{R}$ such that $p^k = p_1 p_2$ with $p_1 \mid x^k$ and $p_2 \mid y^k$. They define $\mathcal{R}$ to be almost Schreier if every nonzero element of $\mathcal{R}$ is almost primal.

A monoid homomorphism $\mathcal{S} : D \rightarrow H$ is called a divisor homomorphism if for any $a, b \in \mathcal{D}$, $\mathcal{S}(a) \mid \mathcal{S}(b)$ in $\mathcal{H}$ implies $a \mid b$ in $\mathcal{D}$. A monoid $\mathcal{H}$ is called inside factorial if there exists a divisor homomorphism $\mathcal{S} : D \rightarrow H$ from a factorial monoid $\mathcal{D}$ such that for every $x \in \mathcal{D}$, there exists an integer such that $x^r \in \mathcal{S}(\mathcal{D})$. An integral domain $\mathcal{R}$ is called inside factorial if its multiplicative monoid $\mathcal{R} = \mathcal{R} \setminus \{0\}$ is inside factorial. (See for example [4], [12] for information on this class of rings and monoid.)

A framework of this article is as per the following. In Section 2, a classification is given of when a $M$-evaluated area $\mathcal{R}$ is nearly Schreier, under the presumption that $\mathcal{R}$ is a root augmentation (see Theorem 2.5), where $\mathcal{M}$ is a without torsion, cancellative monoid. By practicing to the case that $\mathcal{R} = \mathcal{D}[\mathcal{M}]$ is a monoid space, it is appeared in Theorem 2.8 that if $\mathcal{D}[\mathcal{M}]$ is a root augmentation and $\mathcal{M}$ is funnel shaped, at that point $\mathcal{D}[\mathcal{M}]$ is nearly Schreier if and just if $\mathcal{D}$ and $\mathcal{M}$ are nearly Schreier. This is a partner to an aftereffect of Krause [12, Theorem 3.2], which gives the comparing result with "nearly Schreier" supplanted by "inside factorial." In Section 3, a classification is given of when an evaluated space is inside factorial, as far as the relatively primal property (see Theorem 3.1). In Section 4, a sufficient condition is given on the requests $\mathcal{R}$ in a quadratic number ring for $\mathcal{R}[X]$ to be an IDPF ring, by means of the discriminate $\mathcal{S}(\mathcal{d})$ of $\mathcal{S}(\mathcal{d})$. In Section 5, it is demonstrated that if $\mathcal{R} = \mathcal{E}[n x]$ is a request in a quadratic augmentation field $\langle d \rangle$ of $\mathcal{S}$, at that point the conditions $\mathcal{R}[X]$ is IDPF; $\mathcal{R}[X]$ is inside factorial; $\mathcal{R}[X]$ is nearly Schreier; $\mathcal{R}[X]$ is a root augmentation; and
each prime divisor of $n$ additionally separates the
discriminate of the expansion $K/S$; are proportional
conditions. This gives a valuable class of cases of rings
fulfilling the conditions considered here, including
illustrations demonstrating that the root expansion
condition is fundamental in the previously mentioned
Theorems 2.5 and 2.8.

II. ALMOST SCHREIER GRADED
DOMAINS

In this area, let $R$ m $M$ Rm be a M-reviewed space,
where $M$ is a without torsion, cancellative, commutative
monoid, and let $S$ mean the arrangement of nonzero
homogeneous components of $R$. We say that $R$ is gr-
near Schreier if at whatever point $x$ $y$, where $x$, $y$, $S$,
there exists a whole number $k1$ to such an extent that $sk
s1s2$ with $s1 x$ and $s2 y$. Let $x x$, $x$ n be the novel
portrayal of $x$ as a whole of homogeneous components.
At that point, the substance of $x$ is the homogeneous
perfect $C(x) (x1, ..., xn)R$ of $R$.

In Theorem 2.5, we demonstrate a halfway simple
of an outcome [3, Theorem
2.1] on pre-Schreier reviewed areas for when evaluated
spaces are nearly Schreier. Obviously the greater part
of the factorization properties of basic areas considered
in this article has partners for abelian sans torsion
cancellative monoid. See for instance [7]. Since our
monoid are generally composed additively, we frequently
find it advantageous utilize the characteristic preorder
on $M$ which is defined by $x$ if $x$ $y$ for some $z M$. It is
anything but difficult to see that the normal preorder
on $M$ is ant symmetric, and in this manner a halfway request
on $M$ if and just if $M$ is tapered; that is, $x y$ = 0 in $M$
suggests $x = y = 0$.

On the off chance that $M$ is a multiplicative
monoid, $Y M$ and $x M$, we compose $x Y$ if $x$ for every
$Y$. On the off chance that $Y1, Y2 M$, $Y1Y2$ means $\{y1y2
Y1, y2 Y2\}$.

(†): For any nonempty finite subsets $Y1, Y2 \subseteq M$ and $x \in M$
such that $x | Y1Y2$, there exist $z1, z2 \in M$ and an integer
$k \geq k1$ such that $x = z1z2$ with $z1 | Y1$ and $z2 | Y2$.
(II) $M$ is an additive monoid, then in terms of the natural
preorder, this becomes as follows: if $x \leq Y1 + Y2$, there
are $z1, z2 \in M$ and an integer $k \geq k1$ such that $kx = z1 + z2$ with $z1 \leq kY1$ and $z2 \leq kY2$.

Clearly, for any monoid $M$, condition (†) implies $M$
is almost Schreier. In the next lemma, we see that the
converse is true for cancellative monoid.

Lemma 2.1. If $M$ is almost Schreier, then $M$ satisfies (†).

Proof. The proof is by induction on $n = |Y1| |Y2|$ the
case $n = 2$, when $|Y1| |Y2$ is true by definition of almost
Schreier.

Suppose the claim is true for all $n \leq N$ for some $N$
in $\in \mathbb{N}$. To prove the claim for $n = N + 1$, we will assume
that there are nonempty subsets $Y1, Y2 M \subseteq N =
|Y1| + |Y2|$ and $x, y1 \in Y1$ such that $x (Y1 \cup Y2) + Y2$,
and show that there exists a positive integer $k$ such that
$kx = z1 + z2$ for some $z1, z2 \in M$ such that $z1 \leq kY1$
and $z2 \leq kY2$.

By hypothesis, we have the two inequalities,
$x \leq y1 + Y2$ and $x \leq y1 + Y2$.

By the induction hypothesis, the first of these inequalities
implies that there are $w1, w2 \in M$ and an integer $k \geq 1$
such that
$kx = w1 + w2$ with $w1 \leq kY1$ and
$w2 \leq kY2$.

In particular, $kY2 = w2 + Z2$ for some subset $Z2
M \subseteq Z$ with $|Z2| = |Y2|$. From the equality in (2) and
the second inequality in (1), we now get
$w1 + w2 = kx \leq kY1 + kY2 = kY1 + w2 + Z2$.

Since the monoid $M$ is cancellative, (3) implies $w1 \leq kY1$.
We have $w1 \leq kY1$.

(††): For any $x, y \in R$, $s \in S$ and $y \in (s) : R(x)$, then $C(y^k) \subseteq C(x^k)$ for some positive integer $k$.

Lemma 2.2. If $R = m \subseteq M$ $Rm$ is almost Schreier, then $R$
satisfies (††).

Proof. Let $y \in (s) : R(x)$, where $x, y \in R, s \in S$. Then $xy \in (s)$
and so $s | xy$. Thus since $R$ is almost Schreier, $s = s1s2$
with $s1 \mid x^k$ and $s2 \mid y^k$ for some integer $k \geq 1$. Clearly $s1,
\text{ and } s2$ are in $S$. Then $y^k \in (s2)$, a homogeneous ideal.
Thus $C(y^k) \subseteq (s2)$ Similarly, $C(x^k) \subseteq (s1)$. So we have
$C(x^k) \subseteq (s1) \subseteq (s2)$ and therefore, $C(x^k)C(y^k)
\subseteq (s1s2) = (s)$. So $C(y^k) \subseteq (s) : R C(x^k)$.

We use the following result from [15].

Lemma 2.3 ([15, Corollary 3.3]), If $D$ is an integrally
closed domain, $d, a1, ..., an \in D$ for $i = 1, ..., r$, then $d | a1^m \cdots an^m$
wherever $n_i \geq 0$ and $\sum n_i = n$. Then $d \mid a_i$. 

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Lemma 2.4. Suppose $R = mR_m$ is gr-almost-Schreier, that $R$ satisfies $(\dagger\ddagger)$ and that $R \subseteq \mathcal{S}$ is an $\alpha$-root extension. Then every $s \in \mathcal{S}$ is almost prime in $R$.

Proof. Let $s \mid xy$. Then $xy = sr$ for some $r \in R$. Thus $y \in \langle s \rangle : \langle x \rangle$. So, by $(\dagger\ddagger)$, $C(y) \subseteq \langle s \rangle : \langle x \rangle$ for some integer $k \geq 1$. Then $C(s)C(y) \subseteq \langle s \rangle$. So $s$ divides every member of $C(x)sC(y)$. Let $x = x_1 + x_2 + \cdots + x_m$, $x_i \in R_m$, $a_i = a_j$ if and only if $i = j$, and $y = y_1 + y_2 + \cdots + y_m$, $y_i \in R_m$.

Theorem 2.5. Consider the following conditions on $R = mR_m$.

(1) $R$ is almost Schreier.
(2) Every element $s \in \mathcal{S}$ is almost prime in $R$.
(3) $R$ is gr-almost-Schreier and $R$ satisfies $(\dagger\ddagger)$.

Then (1) $\implies$ (2) $\implies$ (3). If $R \subseteq \mathcal{S}$ is a root extension, these equivalent.

Proof. (1) $\implies$ (2): As $R$ is almost Schreier, every element in the saturated multiplicative set $S$ is almost prime in $R$.

(2) $\implies$ (1): By [1, Proposition 2.1], $R_3$ is a GCD-domain, and hence it is almost Schreier [6, Proposition 2.2(a)]. As every $s \in \mathcal{S}$ is almost prime in $R$, it follows by the Nagata type result for almost Schreier domains [6, Theorem 4.3], that $R$ is almost Schreier.

(3) $\implies$ (2): This follows from Lemma 2.4.

Lemma 2.6. If $M$ is conical, $D$ and $M$ are almost Schreier, and $D[M] \subseteq \mathcal{D}[M]$ is a root extension, then $D[M]$ is gr-almost-Schreier and satisfies $(\dagger\ddagger)$.

Proof. Suppose $r, X^0 | r_2 X^{a_2} \cdots r_n X^{a_n}$ in $D[M]$. Then, $r_2 | r_3$ in $D$ and $a_1 \leq a_2 + a_3$ in $M$, where $\leq$ is the natural order on $M$. Since $D$ is almost Schreier, there exists an integer $k_2 \geq 1$ and $w_2, w_3$ in $D$ such that $k_2 a_1 = p_2 + p_3$ and $p_2 \leq k_2 a_2$ and $p_3 \leq k_2 a_3$. From Eqs. (6) and (7), we get $k_3 a_1 = p_2 + p_3$ with $p_2 \leq k_3 a_2$ and $p_3 \leq k_3 a_3$. So in $D[M]$ is almost Schreier, Lemma 2.1 implies that there exists an integer $k_2 \geq 1$ and $w_2, w_3$ in $D$ such that $k_2 a_1 = p_2 + p_3$ and $p_2 \leq k_2 a_2$ and $p_3 \leq k_2 a_3$. Thus, there follows from Lemma 2.4.

We now show that $D[M]$ satisfies $(\dagger\ddagger)$. Let $g \in (rX^a)$, $\gamma \in (D[M])$, and $\gamma \in (D[M])$ is a homogeneous element of $D[M]$. So, $rX^a \gamma \in D[M]$, and hence in $D[M]^{-1}$. So $rX^a \gamma$ divides every element of $(Cfg)(Cfg)$. So $D[M]^{-1}$ is integrally closed, $(Cfg)(Cfg)$, [2, Theorem 3.5(1)]. So $rX^a \gamma$ divides every element of $(Cfg)(Cfg)$ in $D[M]^{-1}$. Let $f = rX^a \gamma + \cdots + rX^a \gamma$ where $a_i = a_j$ if and only if $i = j$, and $g = sX^a \gamma + \cdots + sX^a \gamma$ where $h_i = h_j$ if and only if $i = j$. Thus, $r_i | r_j$ for all $i$ and $j$ in $D$. As $D$ is almost Schreier, Lemma 2.1 implies that there exists a positive integer $k_i \geq 1$ and $w_i | X^a \gamma$ such that $r_i = z w_i$ with $z | X^a \gamma$, and $w_i | X^a \gamma$ for all $i$ and $j$.

Also, $X^a \gamma X^a \gamma$ for all $i$ and $j$. So $a_i \leq a_j$ for all $i$ and $j$. As $M$ is almost Schreier, Lemma 2.1 implies that there exists an integer $k_2 \geq 1$ and $y_i \in M$ such that $k_2 a_i \geq 1 + y_i$, $y_i \in M$ such that $k_2 a_i \geq k_2 a_2$ and $y_i \leq k_2 a_3$ for all $i$ and $j$. Then $r_i | r_j$ for all $i$ and $j$. If $k_i a_1 = k_2 a_2$ and $k_3 a_1 = k_2 a_3$, then $k_i a_1 = k_2 a_2$ and $k_3 a_1 = k_2 a_3$. Thus $k_i a_1 = k_2 a_2$ and $k_3 a_1 = k_2 a_3$. Thus $D[M]$ is integrally closed, $(Cfg)(Cfg)$ in $D[M]$. As $D[M]$ is integrally closed, $(Cfg)(Cfg)$ in $D[M]$ whenever $i \geq 0$ and $\Sigma l_i = k_i a_2$ by Lemma 2.3.
Lemma 2.7. If $M$ is conical and $D[M]$ is almost-Schreier, then $D$ and $M$ are almost Schreier.

Proof. Let $r_1y_1^j$ be an integer $k \geq 1$ and $r_1, r_2 \in D[M]$ such that $r_1 = r_2, r_1y_1^j, r_2y_1^j$. Since the degree of $r$ is 0, it follows that $r_1$ and $r_2$ have degree 0 since $M$ is conical. Thus $r_1, r_2 \in D$. So $D$ is almost Schreier. Now suppose $a \leq a_1 + a_2$. Then $X^a X^{a_1} = X^{a_2}$. As $D[M]$ is a graded domain, there exists an integer $k \geq 1$, and $f_1, f_2 \in D[M]$ such that $X^a = f_2 X^a$ where $f_1 X^{a_1}k$ and $f_2 X^{a_2}$. By [9, Theorem 11.1], $f_1$ and $f_2$ are monomials. So $f_1 = uX^\phi$ and $f_2 = u^{-1}X^{\psi}$. Thus $X^a = uX^\psi = uX_{p+2} + uX_{p+1}X_{p} + \cdots + uX_{q}X_{q-1}$. So $a_k = b_k p + 2$ and $a_k = b_k q + 1$. So $M$ is almost Schreier.

Reference


