International Journal of Scientific Research & Engineering Trends
Volume 4, Issue 4, July-Aug-2018, ISSN (Online): 2395-566X

A Study on Topological Groups and Haar Measures

M.Phil.Scholar Suma Joseph
Department of Mathematics
Prist University, Thanjavur
Tamil Nadu, India

Dr.B.Senthil Kumar
Department of Mathematics
Prist University, Thanjavur
Tamil Nadu, India

Abstract **-Compact right topological groups arise in topological dynamics and in other settings. Following H. Furstenberg's seminal work on distal flows, R. Ellis and I. Namioka have shown that the compact right topological groups of dynamical type always admit a probability measure invariant under the continuous left translations; however, this invariance property is insufficient to identify a unique probability measure (in contrast to the case of compact topological groups). We amplify on the confirmations of Ellis and Namioka to show that a right invariant probability measure on the compact right topological group G exists provided G admits an appropriate system of normal Subgroups, that it is uniquely determined and that it is also invariant under the continuous left translations. Using Namioka's work, we show that G has such a system of subgroups if topological centre contains a countable dense subset, or if it is a closed subgroup of such a group.**

Keywords **– Cantor set, compact group, Haar measure, Lebesgue measure etc.**

I.INTRODUCTION

Topological groups and Haar measures.Our purpose in this section is to describe the connection between two different approaches to theory. The measure theory approach is commonly used in probability theory and so is familiar to most of us. However, the algebraic or linear functional approach is not so familiar. A good reference for the details of both approaches is Segal and Kunze (1978). A brief description of the linear functional approach and its relationship to the measure theory approach follows.

Let X be a locally compact topological space (Hausdorff) for which the topology has a countable base. That is, **X** is a topological space such that:

II. NECESSARY OF TOPOLOGICAL SPACE AND HAAR MEASURE

Definition 1. Topological spaces

If X is a set, a family U of subsets of X defines a topology on X if

(i) $\emptyset \in u,\ X \in u$.

(ii) The union of any family of sets in u belongs to u.

(iii) The intersection of a finite number of sets in U belongs to U. If U defines a topology on X, we say that X is a topological space. The sets in U are called open sets. The sets of the form $X\backslash U,\ U \in u$, are called closed sets. If Y is a subset of X the closure of Y is the smallest closet set in X that contains Y.

Let Y be a subset of a topological space X. Then we may define a topology $\curlywedge_Y$ onY, called the subspace or relative topology, or the topology on Y induced by the topology on X, by taking $\curlywedge_Y = \{Y \cap R \mid R \in \curlywedge\}$

A system B of subsets of X is called a basis (or base) for the topology R if every open set is the union of certain sets in B. Equivalently, for each open set R, given any point $x \in R$, there exists $B \in B$such that $x \in B \subset R$.

Definition 2

A Radon measure is a Borel measure on a Hausdorff locally compact topological space which is finite on compact sets, inner and outer– regular on all open sets.

Definition 3

A Haar measure on a locally compact topological group G is a non–zero Radon measure which is the right translation –invariant,

i.e. $\curlywedge(gE) = \curlywedge(E)$

For any Borel subset E of G and each g ∈ G.

One can similarly define left translation–invariant measures, or bi-invariant translation –invariant measures, which is a combination of both.

Definition4

We say that (G,y, ⋋) is a topological group if (G,y) is a group and (G, ⋋) a topological space such that, writing M(y, x) = y × x and Jy = y^{-1} the multiplication map M : G^2 → G and the inversion map J : G → G is continuous

Definition 5

For G a topological group, there exists a measure $\curlywedge_{Hear}$ or simply⋋,which is left G-invariant. Formulated simplistically, this is a countably additive function $\curlywedge: \{open\ sets\ of\ G\} \to (0,\infty)$With the invariance property$\curlywedge(gA) = \curlywedge(A)$, for all $A \subset G$and$g \in G$. Here gA= {g x:$g \in A\} \subset G$. Moreover, μ is unique up to a scalar multiple. We fix now on the Haar measure⋋. For $A \subset G$ an open set, we may refer to ⋋(A) also as vol (A).

Definition 6

A group G is uni modular if $\curlywedge(g) = 1, \forall \in G$.In other words, a group is unimodularif the Haar measure (by definition left-invariant) is right-invariant as well.

Theorem. 1

A group of the following type is necessarily unimodular:

- Abelian groups
- Compact groups
- SL$(2,\mathbb{R})$

Proof

If G is a compact group, the image ⋋ $(G)$is a compact subgroup of $(0,\infty)$, in particular bounded. Hence ⋋ (G) = $\{1\}$. c) But every element of $sl(2,\mathbb{R})$ is of this form (check!), hence$⋋_*\equiv 0$. Since $SL(2,\mathbb{R})$ is connected this determines ≡ 1 .

Example

For G=$\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix}\subset GL(2,\mathbb{R})$, let $d_{Haar}\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) = \sigma(y,x)dxdy$.

Since$\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} bx & by+a \\ 0 & 1 \end{bmatrix}$,

Left-invariance of Haar measure means $\sigma(ay,ax+ba2dxdy= \sigma y,xdxdy$, hence for $x=1$ and $y=0$ we obtain $\delta(a,b)=a^{-2}$, so the Haar measure is$\frac{dxdy}{y^2}$. To compute the modulus,

Let$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} bx & bx+y \\ 0 & 1 \end{bmatrix}$.

Then $\frac{|a|dxdy}{a^2x^2} = \delta(g)\frac{dxdy}{x^2}$,

Hence $\delta\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} = |a|^{-1}$.

This shows that G is not unimodular. In particular, for g=$\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix}$ we have

$$\int_G \emptyset(h)dh = \int_{R^x}\int_R \emptyset\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right)\frac{dxdy}{x^2}$$

$$\int_G \emptyset(gh)dh = \int_G f(h)dh$$

$$\int_G \emptyset(hg)dh = |a|\int_G \emptyset(h)dh$$

Let B_m the upper triangular sub group of GL (m,ℝ).

Then $dg = \frac{\Pi_{1<i<j<n}dx_{ij}}{\Pi_{i=1}^{m} x_{ii}^{m-i+1}} = \det(g)^{-n-1}\frac{\Pi dx_{ij}}{\Pi x_{ii}^{i}}$,

$$\delta_{B_n}(g) = x_{11}^{n-1}x_{22}^{3-n}\dots x_{mm}^{m-1} = det(g)^{-n-1}\prod_{k=1}^{n} x_{ii}^{2i}$$

Theorem 2

Let G be a topological group. Then

- The map g ↔ g^{-1} is a homeomorphism of G onto itself.
- Fix$g_0\epsilon\, G$.Themaps$g\leftrightarrow g_0 g, g\leftrightarrow g g_0$,and$g\leftrightarrow g_0 g g_0^{-1}$ are homeomorphisms of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G, and let M: $G\rightarrow G/H$be the canonical mapping of G onto G/H. We define a topology⋋*uG/H* on *G/H*, called the quotient topology,by$⋋_{G/H}$ $\{p(⋋)|\ ⋋\in⋋_G\}$.(Here,$⋋_G$is the topology on G).The canonical map p is open (by definition) and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a normal subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map M : G ↔ Gꞌ is a continuous homomorphism of G into G' if M is a homomorphism of groups and M is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H. A topological group G is a locally compact group if G is locally compact as a topological space.

III. APPLICATION TOPOLOGICAL GROUPS AND HAAR MEASURE

Example 1

Take G = R^m with addition as the group operation and the usual topology on R^m. Obviously, 0 $\in R^m$ is the identity and -a is the inverse of a$\in R^m$. This group is commutative, that is, a + b =b +a, for a,b$\in R^m$. This example clearly extends to the case where G is a finite dimensional real vector space with the Euclidean topology

Let G = G$⋋_n$, which is the group of all m X m non-singular real matrices with matrix multiplication as the group operation-commonly called the general linear group. For g∈ G$⋋_n$, g- 1 means the matrix inverse of g so that the identity in G$⋋_n$ is the m X m identity matrix. Let $\varphi_{m,m}$ be the real vector space of m X m real matrices so G$⋋_n\subseteq\ \varphi_{m,m}$ The determinant function det is defined on $\varphi_{m,m}$ and is continuous.

Since G$⋋_n$= $\{a/a\in\varphi_{m,m},\det(a)\neq 0\}$

We see that G$⋋_n$ is an open subset of the Euclidean space $\varphi_{m,m}$·That is, G$⋋_n$ is the complement of the closed set $\{a/a\in\varphi_{m,m},\det(a)=0\}$Thus,G$⋋_n$is a space with the topology inherited from $\varphi_{m,m}$· that the group operations are continuous is not too hard to check.

Example 2.

Let G be the group G$⋋_n$ of m X m lower triangular matrices whose diagonal elements are positive. The group operation is matrix multiplication and G$⋋_n$, is easily shown to be closed under this operation. That g^{-1} ∈G$⋋_n$ for g ∈G$⋋_n$ is most easily established by induction. For g∈G$⋋_n$ partition p as

$$G=\begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}$$

Where g_{11} is (m- 1) X (m- 1) and is lower triangular with positive diagonals,g_{21} is 1 X (m- 1) and $g_{22}\in(0,\infty)$ the inverse of g is

$$\begin{pmatrix} g_{11}^{-1} & 0 \\ -g_{22}^{-1}g_{21}g_{11}^{-1} & g_{22}^{-1} \end{pmatrix}$$

which shows that $g^{-1}\in$ G$⋋_n$The topology for ∈ G$⋋_n$is that obtained by regarding ∈ G$⋋_n$a subset of m(m + 1)/2 dimensional coordinate space and using the inherited topology for ∈ G$⋋_n$Clearly a; is an open subset of this m(m + 1)/2 dimensional vector space. The continuity of the group operations follows from the fact that ∈ G$⋋_n$is a (closed) subgroup of ∈ G$⋋_n$ and hence inherits the continuity of the group operations from ∈ G$⋋_n$.

Example 3

The final example of this chapter concerns $G\curlywedge_n$ $\varphi_{m,m}$.the left and right Haar measures, the modular function and all the multipliers for $G\curlywedge_n$ are derived here. We are going to use the method described. For a $\in G\curlywedge_n$, x has the form.

$$A=\begin{pmatrix} x_{11} & 0 \ldots & 0 \\ x_{21}x_{22} & \ldots . 0 & \\ & . & \\ & . & \\ & . & \\ & . & \\ x_{m1}x_{m2} & & \ldots . x_{mm} \end{pmatrix}$$

With$x_{ii}>$ 0 and $x_{i,j} \in R^m$ i$> j$. Let dx denote Lebesgue measure restricted to the set of such x's in m (m + 1)/2 dimensional space. Define an integral Jon K($G\curlywedge_n$) by

$$J(L_g M)=\int M(x)dx$$

And for g$\in$ G $\curlywedge_n$ consider

$$J(L_g M)=\int M(g^{-1}x)dx$$

With b=$g^{-1}x$ so a=gb, Jacobian of this transformation on m(m + 1)/2 coordinate space is

$$\curlywedge_0(g) = \prod_{i=1}^{m} g_{i,i}^{i}$$

Where $g_{11} \ldots \ldots g_{mm}$ are the diagonal elements g$\in G\curlywedge_n$. Hence dx=$\curlywedge_0 (g)dy$ so that

$$J(L_g M)=\curlywedge_0 (g)J(M)$$

$$v_i(dx) = \frac{dx}{\curlywedge_0 (x)} = \frac{dx}{\prod_{i=1}^{m} x_{i,i}^{i}}$$

is a left Haar measure on $G\curlywedge_n \subseteq$ $\varphi_{m,m}$. The modulus of $G \curlywedge_n$ is computed. Thus, consider

$$\int M(xg^{-1})\, v_i(dx) = \int M(xg^{-1}) \frac{dx}{\curlywedge_0 (x)}$$

And let b= xg^{-1} so that x=bg. Jacobian of this transform is

$$\curlywedge_1(g) = \prod_{i=1}^{m} g_{ii}^{m-i+1}$$

Where$g_{11} \ldots \ldots g_{mm}$ are the diagonal elements of g. making this change of variables in the above integral yields

$$\int M(xg^{-1})\, v_i(dx) = \int M(y) \curlywedge_1 (g) \frac{dx}{\curlywedge_0 (x)}$$

$$\frac{\curlywedge_1 (g)}{\curlywedge_0 (g)} \int M(y) \frac{dx}{\curlywedge_0 (x)}$$

$$\frac{\curlywedge_1 (g)}{\curlywedge_0 (g)} \int M(y)\, v_i(dx)$$

Thus by definition, the modulus of $G\curlywedge_n$ is

$$\Delta(g) = \frac{\curlywedge_1 (g)}{\curlywedge_0 (g)} = \prod_{i=1}^{m} g_{ii}^{m-2i+1}$$

So a right Haar measure is

$$v_i(dx) = \frac{1}{\Delta(g)} v_i(dx) = \frac{dx}{\prod_{i=1}^{m} g_{ii}^{m-i+1}}$$

Here is a characterization of the multipliers on $G\curlywedge_n \subseteq$ $\varphi_{m,m}$

IV.CONCLUTION

We have studied the topological group's structures on the Cantor set R and shown that any such structure has an "equivalent" Abelian group structure, in the sense that the Haar measures are the same. We also showed any representation of "g" as an abelian group admits a continuous mapping onto the unit interval sending Haar measure to Lebesgue measure.

REFERENCES

[1]. Hewitt, E.; Ross, K.A. Abstract Harmonic Analysis; Springer-Verlag: Heidelberg, Germany, 1963.
[2]. Rudin, W. Fourier Analysis on Groups;Wiley: New York, NY, USA, 1962.
[3]. Husain, T. Introduction to Topological Groups; W.B. Saunders Co.: Philadelphia, PA, USA, 1966.
[4]. Pontryagin, L.S. Topological Groups; Princeton University Press: Princeton, NJ, USA, 1946.
[5]. Montgomery, D.; Zippin, L. Topological Transformation Groups; Interscience Publishers: New York, NY, USA, 1955.
[6]. Morris, S.A. Pontryagin Duality and the Structure of Locally Compact Abelian Groups; Cambridge University Press:Cambridge, UK, 1977.
[7]. Markov, A.A. On free topological groups. Dokl. Akad. Nauk SSSR **1941** 31, 299–301.
[8]. Markov, A.A. On free topological groups. Izv. Akad. Nauk SSSR Ser. Mat. **1945**, 9, 3–64.
[9]. Graev, M.I. Free topological groups. Izv. Akad. Nauk SSSR Ser. Mat. **1948**, 12, 279–324.
[10].Hofmann, K.H. The Lie Theory of Connected Pro-Lie Groups; European Mathematical Society: Zürich, Switzerland, 2007.
[11].Hofmann, K.H.; Morris, S.A. The Structure of Compact Groups, 3rd ed.; Revised and Augmented; VerlagWalterde Gruyter Berlin: Berlin, Germany, 2013.
[12]. Graev, M.I. The theory of topological groups I. Uspekhi Mat. Nauk 1950, 5, 3–56.
[13]. Nakayama, T. Note on free topological groups. Proc. Imp. Acad. Tokyo 1953, 19, 471–475.
[14]. Kakutani, S. Free topological groups and infinite direct product of topological groups. Proc. Imp. Acad. Tokyo1944, 20, 595–598.
[15]. Mal'tsev, A.I. Free topological algebras. Izv. Akad. Nauk SSSR Ser. Mat. 1957, 21, 171–198.