A Study on Topological Groups and Haar Measures

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Abstract - Compact right topological groups arise in topological dynamics and in other settings. Following H. Furstenberg's seminal work on distal flows, R. Ellis and I. Namioka have shown that the compact right topological groups of dynamical type always admit a probability measure invariant under the continuous left translations; however, this invariance property is insufficient to identify a unique probability measure (in contrast to the case of compact topological groups). We amplify on the confirmations of Ellis and Namioka to show that a right invariant probability measure on the compact right topological group \( G \) exists provided \( G \) admits an appropriate system of normal Subgroups, that it is uniquely determined and that it is also invariant under the continuous left translations. Using Namioka's work, we show that \( G \) has such a system of subgroups if topological centre contains a countable dense subset, or if it is a closed subgroup of such a group.

Keywords – Cantor set, compact group, Haar measure, Lebesgue measure etc.

I. INTRODUCTION

Topological groups and Haar measures. Our purpose in this section is to describe the connection between two different approaches to theory. The measure theory approach is commonly used in probability theory and so is familiar to most of us. However, the algebraic or linear functional approach is not so familiar. A good reference for the details of both approaches is Segal and Kunze (1978). A brief description of the linear functional approach and its relationship to the measure theory approach follows.

Let \( X \) be a locally compact topological space (Hausdorff) for which the topology has a countable base. That is, \( X \) is a topological space such that:

II. NECESSARY OF TOPOLOGICAL SPACE AND HAAR MEASURE

Definition 1. Topological spaces

If \( X \) is a set, a family \( U \) of subsets of \( X \) defines a topology on \( X \) if

(i) \( \emptyset \in U \), \( X \in U \).

(ii) The union of any family of sets in \( U \) belongs to \( U \).

(iii) The intersection of a finite number of sets in \( U \) belongs to \( U \). If \( U \) defines a topology on \( X \), we say that \( X \) is a topological space. The sets in \( U \) are called open sets. The sets of the form \( X \setminus U \), \( U \in U \), are called closed sets. If \( Y \) is a subset of \( X \) the closure of \( Y \) is the smallest closed set in \( X \) that contains \( Y \).

Let \( Y \) be a subset of a topological space \( X \). Then we may define a topology \( \gamma \) on \( Y \), called the subspace or relative topology, or the topology on \( Y \) induced by the topology on \( X \), by taking \( \gamma = \{ Y \cap R \mid R \in \} \).
Theorem 1
A group of the following type is necessarily unimodular:
- Abelian groups
- Compact groups
- SL(2, ℝ)

Proof
If G is a compact group, the image x(G) is a compact subgroup of (0, ∞), in particular bounded. Hence x(G) = {1}. c) But every element of sl(2, ℝ) is of this form (check!), hence x, ≡ 0. Since SL(2, ℝ) is connected this determines ≡ 1.

Example
For G = [0, 1]^2 ⊆ GL(2, ℝ), let d_Haar(x 0 y 0) = σ(y, x)dx dy.

Since \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} bx + a \\ y \end{bmatrix},

Left-invariance of Haar measure means σ(ay, ax + ba2dx dy = ax, ax dy), hence for x = 1 and y = 0 we obtain δ(a, b) = a^{-2}, so the Haar measure is dx dy = x^{-2}. To compute the modulus,

\[
\delta \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} = \frac{|b|^2}{x^2}.
\]

This shows that G is not unimodular. In particular, for

\[
g = \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix}
\]

we have

\[
\int_G \phi(h)dh = \int_{Y} \int_{X} \phi \begin{bmatrix} x \\ y \end{bmatrix} \frac{dxdy}{x^2}.
\]

\[
\int_{X} \int_{Y} \phi(g)dh = \int_{X} \int_{Y} \phi \begin{bmatrix} x \\ y \end{bmatrix} \frac{dxdy}{x^2}.
\]

Hence

\[
\delta \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} = |a|^{-1}.
\]

Theorem 2
Let G be a topological group. Then
- The map g ↦ g^{-1} is a homeomorphism of G onto itself.
- Fix g ∈ G. The map φ ↦ g φ g^{-1} = g φ g^{-1} is a homeomorphism of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G, and let M: G ↦ G/H be the canonical mapping of G onto G/H. We define a topology by xG/H = {p(φ) | φ ∈ xG}, (Here xG is the topology on G). The canonical mapping p is open (by definition) and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a normal subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map M: G ↦ G' is a continuous homomorphism of G into G' if M is a homomorphism of groups and M is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H. A topological group G is a locally compact group if G is locally compact as a topological space.

III. APPLICATION TOPOLOGICAL GROUPS AND HAAR MEASURE

Example 1
Take G = R^m with addition as the group operation and the usual topology on R^m. Obviously, 0 ∈ R^m is the identity and -a is the inverse of a ∈ R^m. This group is commutative, that is, a + b = b + a, for a, b ∈ R^m. This example clearly extends to the case where G is a finite dimensional real vector space with the Euclidean topology.

Let G = G_λ_n, which is the group of all m X m non-singular matrixes with matrix multiplication as the group operation–commonly called the general linear group. For g ∈ G_λ_n, g_1 = 1 means the matrix inverse of g so that the identity in G_λ_n is the m X m identity matrix.

Let φ_m,m ∈ φ_m,m. The determinant function det is defined on φ_m,m and is continuous.

Since G_λ_n = {a/a ∈ φ_m,m, det(a) ≠ 0} we see that G_λ_n is an open subset of the Euclidean space φ_m,m. That is, G_λ_n is the complement of the closed set {a/a ∈ φ_m,m, det(a) = 0}. Further, G_λ_n is a space with the topology inherited from φ_m,m that the group operations are continuous is not too hard to check.

Example 2
Let G be the group G_λ_n of m X m lower triangular matrices whose diagonal elements are positive. The group operation is matrix multiplication and G_λ_n is easily shown to be closed under this operation. That is, g_1 ∈ G_λ_n for g ∈ G_λ_n is most easily established by induction. For g ∈ G_λ_n partition p as

\[
G = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}
\]

Where g_{11} is (m-1) X (m-1) and is lower triangular with positive diagonals, g_{21} is 1 X (m-1) and g_{22} ∈ (0, ∞) the inverse of g is

\[
\begin{pmatrix} g_{11}^{-1} & 0 \\ -g_{21}^{-1}g_{12}^{-1}g_{21}^{-1}g_{22} \end{pmatrix}
\]

which shows that \(g^{-1} \in G_\lambda_n\). The topology for \(\in G_\lambda_n\) is that obtained by regarding \(\in G_\lambda_n\) as a subset of \(m(m+1)/2\) dimensional coordinate space and using the inherited topology for \(\in G_\lambda_n\). Clearly a is an open subset of this \(m(m+1)/2\) dimensional vector space. The continuity of the group operations follows from the fact that \(\in G_\lambda_n\) is a (closed) subgroup of \(\in G_\lambda_n\) and hence inherits the continuity of the group operations from \(\in G_\lambda_n\).
Example 3
The final example of this chapter concerns $G \times_n \varphi_{m,m}$, the left and right Haar measures, the modular function and all the multipliers for $G \times_n$ are derived here. We are going to use the method described. For $a \in G \times_n$, $x$ has the form.

\[
A = \begin{pmatrix}
  x_{11} & 0 & \ldots & 0 \\
  x_{21} & x_{22} & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{m1} & x_{m2} & \ldots & x_{mm}
\end{pmatrix}
\]

With $x_i > 0$ and $x_{ij} \in R^m i > j$. Let $dx$ denote Lebesgue measure restricted to the set of such $x$'s in $m + 1/2$ dimensional space. Define an integral Jon $K(G \times_n)$ by

\[
\int (L_x M) = \int M(x) dx
\]

And for $g \in G \times_n$, consider

\[
\int (L_g M) = \int M(g^{-1}x) dx
\]

With $b = g^{-1}x$ so $a = gb$, Jacobian of this transformation on $m(m + 1)/2$ coordinate space is

\[
\lambda_0 (g) = \prod_{i=1}^{m} g_{1,i}^j
\]

Where $g_{11}, \ldots, g_{m m}$ are the diagonal elements $g \in G \times_n$. Hence $dx = \lambda_0 (g) dy$ so that

\[
\int (L_y M) = \lambda_0 (g) \int (L_g M)
\]

\[
v_1 (dx) = \frac{1}{\Delta(g)} \lambda_0 (g) v_1 (dx) = \frac{dx}{\prod_{i=1}^{m} g_{1,i}^{m-i+1}}
\]

Here is a characterization of the multipliers on $G \times_n \subseteq \varphi_{m,m}$

IV. CONCLUSION
We have studied the topological group’s structures on the Cantor set $R$ and shown that any such structure has an “equivalent” Abelian group structure, in the sense that the Haar measures are the same. We also showed any representation of “g” as an abelian group admits a continuous mapping onto the unit interval sending Haar measure to Lebesgue measure.

REFERENCES