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Topological Groups and Different Types of Measures

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Abstract **– Applying Haar's Measure theory to probability theory to understand and generalize the uniqueness of invariant measure in the situation of risk and uncertainty. If T is finite topological group and X be subgroup of T then μ(X)|μ(T).**

Key words**- Invariant measure, Topological group, Haar's Measure, Borel measure etc.**

I. INTRODUCTION

The notion of support of a Borel measure on a metric space can be considered in two meanings. In a more narrow sense, we call the set a support of a measure if it is the smallest closed set which complement has measure zero. In a wider meaning, any set with full measure is called a support. In this paper we will use the second, broader, sense. Taking a countable product of uniform two-points distributions, we obtain an invariant probability measure on the Cantor cube (Cantorset).

Let us describe this construction in a more general case, taking the unit interval instead of the Cantor cube. Fix p ∈(0, 1) and consider a probability space (Ω, F, Pp) with Ω = {0, 1}, F = 2Ω, Pp ({1}) = p. Let µp denotes the product of measures Pp on countable product of (Ω, F), called the Bernoulli measure.By µp we denote the Borel measure on [0, 1] induced by the random variable F : 2N (xi) → ∞ i=1 xi/2i . It is well known ([5]) that µp is a continuous probability measure on [0, 1], positive on each nonempty open interval, and µ 1 2 is equal to the Lebesgue measure λ. We will considersets

Ap = x ∈[0, 1) : limn→∞ x1 + ... + xn n =

where 0.x1x2x3... denotes the binary expansion of x with an infinite number of zeros. The measures µp and their distributions, as well as the sets Ap, were considered and used indifferent areas of mathematics (see for example [5], [8] and [1]). The classic Borel's result from 1909 ([9]) states that the set of simple normal numbers in base 2 A1 2 = x ∈[0, 1) : limn→∞ x1 + ... + xn n = 1 2 has full Lebesgue measure on [0, 1]. An analytic proof of this fact is long and complicated. However, using the Strong Law of Large Numbers, we can easily check that the set Ap is a support of the measure µp (in this way Billingsley argues in [5]). It is clear that for different p1, p2 ∈(0, 1) measures µp1 and µp2 are mutually singular. Since λ A1 2 = 1 and λ A1 2 + t > 0, the set A1 2 ∩ A1 2 + t is nonempty for any t ∈[0, 1). Consequently, each number t from the interval [0, 1) can be represented as a difference of two numbers x and y with limn→∞ x1+...+xn n = limn→∞ y1+...+yn n = 1 2 . Analogously, since the set A1 2 ∩ t − A1 2 is nonempty, each number t from the interval (0, 1] can be represented as a sum of two number swith density of one's equalto12.Inother words [0,1)⊂ A1 2 − A1 2 and (0, 1] ⊂A1 2 + A1 2 . In our paper we discuss the properties of algebraic sums and algebraic differences of sets Ap. The paper is organized as follows. In the first part we show that there exists p such that the set Ap + Ap (and even Ap + Ap + Ap) has an empty interior, although [0, 1) ⊂Ap − Ap. More precisely, if 1 4 ≤ p ≤ 3 4 then [0, 1) ⊂Ap − Ap, and if p < 1 3 then int (Ap + Ap + Ap) = ∅. For the convenience we will consider the interval [0, 1) as the circle – a compact topological group.

In proofs we will mainly use some properties of finite binary sequences described in [12]. Let us note that we can also consider the measures µp and their supports Ap as the Borel measures and Borel sets in the Cantor group C. However, in this case the situation is trivial: we have A1 2 − A1 2 = A1 2 + A1 2 = C and int (Ap − Ap) = int (Ap + Ap) = ∅for p = 1 2.

Invariant Measure is an important tool in many areas of Mathematics, for example the uncertainty principle related to Probability presented in (R.M Dudley, 2002) does not include any statement about an invariant measure, but the measure plays an important role in proving the theorem of probability as will be shown in this paper the probability structure also give rise to an invariant measure of probability distributions. It is interesting to see how the measure theory can be generalized to probability distribution

There is a well-known decomposition theorem [8] which states that the real line can be expressed as the disjoint union of a Lebesgue null set and a set of first category. Although the real line is neither a set of first category nor a set of measure zero (which means that it is small neither in the sense of measure

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nor in the sense of category), the result is significant because for each of two component sin whose disjoin tunionitc and expresse dissmallin one sense or poles for any z_0 ∈D:There exists z_0 ∈D such that P(z) -α (z_0) has at most two distinct zeros and α(z) is nonconstant. Assume thatβ_0 is the zero of P(z) - $\alpha(z_0)$ with multiplicity p and that the multipliers l and k of zeroes of f(z) - (β_0) and f(z)-$\beta_0$and α(z) -α (z_0) at z_0, respectively, satisfy k ≠lp for all f∈ G.but not in the other. Hence, we refer to some generalizations of the above theorem.

Theorem 1 If X is a separable metric space, then a decomposition of X into a μ-null set and a set of first category exists for each σ-finite diffused Borel measure μ on X.However, reparability and σ-finiteness may be replaced by comparatively less stringent requirements.

We say that a cardinal is of measure zero if every finite non-atomic (or diffused) measure defined for all subsets of any set of that cardinality vanishes identically [9]. Any cardinal less than one of measure zero is of measure zero and also all cardinals less than the first weakly inaccessible cardinal are of measure zero.

This concept of a measure zero cardinal has been used earlier to derive many interesting results such as the measure analogue of the Banach category theorem [9] and also an invariant extension of the Lebesgue measure [5]. A measure μ is called semifinite if every set of infinite measure contains a set of positive finite measure. With these two definitions, we now state the following generalization of the above theorem

Theorem 2. Let X be a metric space having a base of measure zero cardinal, and let μ be a diffused (or non-atomic) Borel measure on X such that: (i) μ is semifinite, (ii) every set of measure zero is contained in a Gδ set of measure zero.

Then X can be expressed as the disjoint union of a set of measure zero and a set of first category. The above theorem does not hold for general topological spaces because there exist a normal bicompact and totally disconnected space and a finite Borel measure μ on it such that μ(E) = 0 if and only if E is a set of first category [8]. Theorem2

In a topological space X having a countable dense subset Y and whose every one-element set {x} is Gδ, there exists a Gδ set E ⊇Y such that μ(E) = 0 for every σ-finite, non-atomic Borel measure μ on X. In this theorem, the complement of E in X is certainly a first category set. So, any separable topological space subject to such restrictions as stated above admits decomposition into a μ-null set and a set of first category. Since in any locally compact Haus do rff Space, first countability implies that every one-element set {x} is Gδ, every locally compact Hausdorff first countable space satisfies the conditions of the theorem. Such spaces are not necessarily metrisable, as may be observed from the example of Helley space [6], which constitutes the family of all no decreasing functions from the unit interval [0, 1] into itself with the topology induced by the product topology on [0, 1] [0,1].

Throughout this paper, G denotes a locally compact Hausdorff topological group with e as the identity element, S1 is the σ-ring generated by compact subsets of G (see [3]), S is the σ-ring generated by S1 and subsets of sets in S1 of μ-measure zero, where μ is a non-zero, σ-finite, diffused (this property is equivalent to the no discreetness of the group) regular left Haar measure on S.

The outer measure of anysetE⊆ induced by μis given by μ*(E)=inf{μ(F): E ⊆F ∈S}. The outer measure μ *is σ-finite on E if E ⊆ ∪ ∞ n=1 En where μ *(En) < ∞ for each n. It is called totally σ-finite if it is σ-finite on G. We use the standard notation L 1 (G) for the class of all real-valued μ-integrable functions on G for which ∫G f dμ is finite. The class L 1 (G) is a topological space under the standard norm ‖f‖1 = ∫G |f| dμ. It is also supposed thatour topological group G is endowed with a system of sets which we call the Vitali system

Definition 1 An invariant is something that does not change under a set of transformation. (John Von Neumann, 1999).

Definition 2 Topological group is group with the group operations (binary and inverse) are continuous. (L.S. Pontryagin, 1958).

Definition3 A left Haar's measure μ on topological group T is Radon measure which is invariant under left translation ie. μ (tX) =μ (B) for all t of T, A right measure μ on topological group T is a radon measure which is invariant under right translationie.μ(Xt))=μ(X) for all t belongs to T. (Halmos, P., 1974). Assumption: Topological group is finite. Writing this paper we wonder if general properties described in section 2 are new. We found in the literature only results obtained by Prokaj in [19].

The referee turned our attention to the recent extensive surveys by Ostaszewski ([18]), and Bingham and Ostaszewski ([6], [7]). In these papers the authors reminded a very interesting old paper by Simmons ([20]), containing characterizations of absolute continuity and singularity with respect to the Haar measure, very similar to our ones.Lemma

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II. THE MEASURE OF A SUBGROUP OF AN INVARIANT FINITE TOPOLOGICAL GROUP IS INVARIANT.

Proof: Let T be invariant finite topological group μ(aT)=μ(Ta)=μ(T) by Haar measure let 'e' be the identity element of T'e' belongs to X since X is the subgroup of T μ(eX)= μ(X)μ(Xe)=μ(X) μ(eX)=μ(Xe)=μ(X).

The measure of subgroup is invariant

Theorem1

Measure of a Subgroup of a Finite Topological Group Divides' The Measure of The group

Proof: Let X Is Sub Group Of T (X≤T) , Here X,T Are Finite.

- If X=T nothing tobeproved.
- If X≠T letμ(X)=m and μ(T)=n

we know that every right Haar measure of X in T has same left Haar's measure of X in T. also since

$$\Rightarrow (A \cap X_i) \subseteq X_i$$

μ(Xe)=μ(X) , since X is right Haar measure of X in T.similarly μ(Xa), μ(Xb), μ(Xc), μ(Xd),……….. are right Haar's measures of X in T, then

μ(Xa)=μ(Xb)=μ(Xc)=μ(Xd)=…………..= μ(X)=m

let the number of distinct right Haar's measure of X in T be 'k' All right measures of disjoint sets and induce a partition of T

μ(T)=μ(Xa)+μ(Xb)+μ(Xc)+μ(Xd)+…………..[ktime]n = km

$$\Rightarrow k = \frac{n}{m} \quad \frac{\mu(T)}{\mu(X)} \quad \mu_p \quad \frac{1}{\mu(X)} \quad \text{……………….(1)}$$

Therefore μ(X) divides μ(T)This can also be proved by taking right Haar's measure of X in T.If we extend this idea to probability [6] then the equation (2) is called probability of X in T.is the probability of and it is called probabilistic measure, denoted by $\frac{1}{k}$

(X). Let be measurable in T,and let A bear bitrary

$$\therefore \frac{1}{k} \Rightarrow (A \cap X_i) \subseteq X_i \quad \subseteq \bigcup_{i=1}^{n} X_i$$

sub group of T. A is measurable is also measurable in T.

$$\Rightarrow \quad \subseteq \bigcup_{i=1}^{n} X_i$$

By the above $\mu(A \cap X_i)$ divides $\mu(X_i)$

$$\Rightarrow \frac{\mu(X_i)}{\mu(A \cap X_i)} = q_i \quad \text{(say) or}$$

$$\frac{1}{q_i} = \frac{\mu(A \cap X_i)}{\mu(X_i)} \text{…………} \quad (2)$$

And from (1) $\frac{1}{k_i} = \frac{\mu(X_i)}{\mu(T)}$ ……….(3)

From (2), (3) =

$$\frac{\mu(A \cap X_i)}{\mu(X_i)} \quad \mu(X_i) \quad \text{……….(4)}$$

$$\frac{\mu(A \cap X_i)}{\mu(X_i)} \quad \mu(X_i) \text{…………….(5)}$$

By the above $\mu(A \cap X_i)$ divides $\mu(A)$

this is the probabilistic measure of X_i after A.

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