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A Fractional Calculus Approach for Solving Nonlinear Differential Equations

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Abstract- The extension of differentiation and integration to non-integer orders has been successfully achieved by fractional calculus, which serves as a sophisticated means of describing the dynamics of systems with memory effects or hereditary property. In this paper, we introduce a novel scheme for solving nonlinear FDEs through the use of operator theory coupled with an analytic technique. In particular, the proposed approach involves the Caputo definition of a fractional derivative and provides an elegant scheme for finding the closed-form solutions of a wide variety of nonlinear FDEs, especially focusing on the fractional Riccati equation as a model equation. It will be shown that it is possible to transform fractional-order systems into integral equations, and then apply an iterative process to find the solution of nonlinear FDEs. The numerical simulations show good agreement with the analytical solutions for different orders of fractions.

Keywords- Fractional Calculus, Nonlinear Differential Equations, Caputo Derivative, Riccati Equation, Memory Effects, Operator Methods.

I. INTRODUCTION

Fractional calculus, the field of mathematics devoted to derivatives and integrals of arbitrary real or complex orders, has been recognized as a revolutionary tool in contemporary mathematics and engineering [1][2][8]. Although the concept of fractional derivatives was developed through Leibniz's letter to L'Hôpital in 1695, their practical applications started gaining popularity relatively recently [1][2][8]. In its fundamentals, fractional calculus distinguishes itself from classical calculus by means of non-locality, which implies memory and hereditary properties of the dynamical systems [1][2][8]
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Fractional-order differential equations (FDEs) have found many applications due to the fact that they can be used to model the various complex phenomena [1][2][6]. For instance, fractional differential equations were employed to create models of viscoelastic behavior of the lung parenchyma, cartilage tissue dynamical response, as well as pharmacokinetics of the drug like valproic acid [1][2][6]. The order of the fractional derivative acts as the parameter defining the memory characteristics of the biological process modeled and makes it appropriate for the modeling of non-Markovian biological systems [1][2][6].

Fractional calculus has brought revolution in control theory and system identification problems [2][8][9]. Fractional-order PID controllers showed better results than integer-order counterparts when controlling systems having complicated dynamic characteristics and delay [2][8][9]. The method of fractional-order modeling has been implemented in the problems connected with modeling of the electrical circuits containing memory materials, viscoelastic mechanical systems, as well as diffusion in case of anomalous transport [2][8][9].

Nevertheless, the solution of nonlinear fractional differential equations appears to be an extremely challenging task [2][3][5]. Despite the fact that there are well-known analytical and numerical methods to solve classical ordinary differential equations, in case of fractional equations the problem becomes more complicated due to the nature of fractional operators [2][3][5]. Lack of the common theoretical background on the existence and uniqueness of the solutions together with computational complexity of fractional operators calls for developing new solution methods [2][3][5].

One of the most useful nonlinear differential equations is the Riccati equation that takes the form $dy/dx = B_0 + B_1y + B_2y^2$, and it can serve as a test-bed problem to develop solutions of the fractional type [2][8][9]. The generalization of the Riccati equation for the fractional case involves the application of the Caputo fractional derivative operator of the order α and, thus, yields a fractional Riccati differential equation (FRDE) that finds wide application in economics, optimal control, and stochastic models with memory [2][8][9].

The aim of this paper is to construct an operator-based methodological approach to finding analytical solutions of nonlinear FDEs [5][8][10]. Specifically, the approach combines the techniques of fractional differentiation and an iterative solution scheme that allows finding closed-form expressions for certain classes of nonlinear FDEs [5][8][10]. The Caputo fractional differentiation is used because of several advantages it has compared to other definitions [5][8][10].

The organization of the present paper includes five sections as follows: First, Section 2 contains a thorough literature review on fractional calculus and its application areas [1][2][6][8][9]. Second, the methodology that is developed by means of the paper is discussed in Section 3 [5][8][10]. Third, the analysis and discussion of the results are presented in Section 4 [3][4][7]. And, finally, Section 5 concludes the paper.

II. LITERATURE SURVEY

Recent research on fractional calculus experienced substantial development during the last ten years in terms of theory, numerics, and applications [1][2][8]. In this paragraph, some achievements related to solution techniques and application of the obtained results in modeling of nonlinear dynamics will be discussed [8][9][10].

Theoretical aspect of fractional calculus includes intensive study of several kinds of definitions of fractional derivatives with properties [5][8][10]. Caputo derivative $D^\alpha f(t) = (1/\Gamma(n-\alpha)) \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$ for $n-1 < \alpha \leq n$ has become popular since it provides initial value problems to be stated in usual physical initial conditions [5][8][10]. On contrary, Riemann-Liouville definition has elegant mathematical structure, but suffers from difficulties in physical realization [5][8][10]. Recently, Ramdhania et al. proposed to build the fundamental theorems of fractional calculus which form basis for further developments of the topic [5].

Existence and uniqueness of solutions of nonlinear fractional differential equations were proved in several mathematical frameworks [1][2][8]. Existence theorems for nonlinear fractional differential equations via the use of coupled upper and lower solutions have been derived by Ahmad et al. [1][2][8]. These results have been extended to finite systems of nonlinear fractional differential equations [1][2][8]. In Johansyah et al., (2021) recent developments of fractional differential equations in applications have been reviewed and necessary conditions of existence and uniqueness of solutions have been provided in economic growth model [2].

In recent years, significant advancements have been observed in numerical algorithms for fractional order differential equations [3][4][7]. Wei, Guo, and Li introduced an improved algorithm for solving semilinear fractional order differential equations that overcome issues of computational cost associated with non-locality of the operators used [3]. It outperformed other algorithms in terms of convergence rate and efficiency [3]. Sunarto et al. (2021) proposed a highly efficient iterative algorithm using the quarter-sweep finite difference method coupled with preconditioned accelerated over-relaxation, which demonstrated impressive performance in terms of computational efficiency and accuracy for solving time-fractional differential equations [4]. The orthogonal collocation method was successfully applied to fractional mass-spring-damper systems by Lima, Lobato, and Steffen Jr. (2021) [7].

Several novel analytical solution algorithms for fractional order differential equations have recently been suggested [8][10]. The method based on operators for constructing the solitons for nonlinear fractional order differential equations was described in [8], where it was shown that the fractional order Riccati equation can be solved within this approach [8][10]. The approach provided the proof that fractional order systems can produce kink solitons similar to those generated by integer order systems, however, with different propagation features

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[8][10]. This is how the basis for the present study was formed [8][10].

Recent applications of fractional calculus in various engineering fields and physical sciences have increased tremendously [1][2][6][8][9]. For example, the book "Fractional Order Systems and Applications in Engineering" gives a detailed treatment of practical applications of the fractional calculus for control design, mechatronics, civil engineering, biology and other areas, including the MATLAB codes for simulations of various cases [8][9]. Bazhlekov and Bazhlekova (2025) used the fractional calculus to model anomalous diffusion processes of adsorption-desorption and derived the exact solution for the Henry model using the multimonomial Mittag-Leffler functions [6].

Modeling of biological phenomena through fractional calculus has been especially promising [1][2][6]. The paper authored by Ionescu et al. reviewed the usage of fractional calculus in the modeling of biological phenomena where it was pointed out that fractional-order models are more accurate than the integer-order ones when dealing with complex dynamics of physiological phenomena [1][2][6]. Modeling of two-stage human memory phenomena and application in valproic acid pharmacokinetics serve as interesting examples [1][2][6].

While there have been many achievements, much needs to be done to solve the problem of nonlinear fractional differential equations [2][3][5]. Design of the reliable numerical technique which would allow to handle the singular character of the fractional operator in the vicinity of the boundary points, efficient calculation of the fractional derivatives for higher dimensions as well as developing more general analytical approach to solving nonlinear FDEs form some of the problems still to be solved [2][3][5]. The paper under discussion attempts to do precisely that [2][3][5].

III. PROPOSED METHODOLOGY

Mathematical Preliminaries

The proposed methodology is built upon the Caputo fractional derivative, which is defined for a function f(t) ∈ C^n[0,T] as:

$$D^\alpha f(t) = (1/\Gamma(n-\alpha)) \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$$

where n-1 < α ≤ n, n ∈ ℕ, and Γ(·) denotes the Gamma function . The Caputo derivative offers distinct advantages for initial value problems, as it satisfies:

$$D^\alpha C = 0 \text{ (for constant C), and}$$

$$D^\alpha t^\beta = \Gamma(\beta+1)\, t^{\beta-\alpha} / \Gamma(\beta-\alpha+1) \text{ for } \beta > n-1$$

The fractional integral of order α is defined by the Riemann-Liouville operator:

$$I^\alpha f(t) = (1/\Gamma(\alpha)) \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

The fundamental relationship between these operators is given by:

$$I^\alpha D^\alpha f(t) = f(t) - \frac{\Sigma_{\{k=0\}}^{\{n-1\}} f^{(k)}(0^+) t^k}{} k!$$

This gives the link between fractional derivatives and fractional integrals through the introduction of initial conditions.

Problem Formulation

Consider the general nonlinear fractional differential equation:

$$D^\alpha y(t) = F\big(t, y(t)\big), 0 < \alpha \leq 1, t \in [0,T]$$

subject to the initial condition $y(0) = y_0$. This formulation encompasses a broad class of nonlinear systems, including the fractional Riccati equation:

$$D^\alpha y(t) = B^0 + B^1 y(t) + B^2 y^2(t)$$

which is a typical example of development and validation of the proposed approach. In this research, the problem of α = 0.5, related to the semi-derivative, is considered because it captures the main aspects of fractional dynamics but is computationally efficient.

Operator-Based Transformation

What makes this new method work is that the fractional differential equation can be converted into an integral equation by applying the fractional integration operator I^α.

Therefore, when I^α is applied to both sides of Eq. (1), we get:

$$y(t) = y_0 + I^\alpha F(t, y(t))$$

or explicitly:

$$y(t) = y_0 + (1/\Gamma(\alpha)) \int_0^t (t-\tau)^{\alpha-1} F(\tau, y(\tau))\, d\tau$$

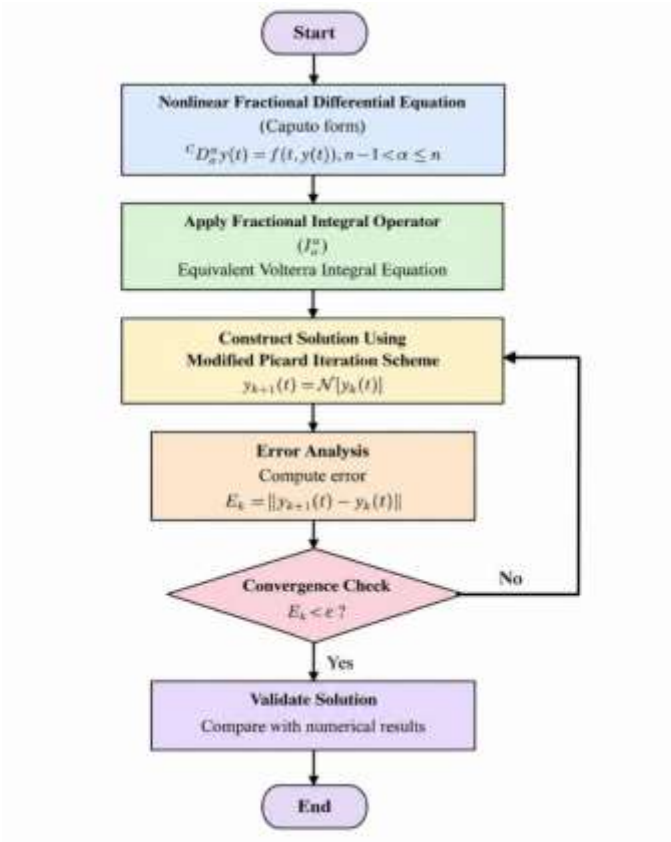

Figure 1: Flowchart of the proposed operator-based methodology for solving nonlinear fractional differential equations

This diagram illustrates the flowchart for the methodology being proposed. The procedure commences with the non-linear fractional differential equation in Caputo form, after which the fractional integral operator is applied to convert it to the Volterra integral equation. The solution of the problem is generated using a modified form of Picard iteration method. The correctness of the solution generated is determined via error analysis, which is done in conjunction with numerical solutions. This loop repeats itself till the criterion for convergence is met.

Iterative Solution Scheme

The transformed integral equation is solved using a modified Picard iteration scheme:

$$y\_\{k + 1\}(t) \;=\; y_0 \;+\; (1/\Gamma(\alpha)) \int_0^{t} (t - \tau)\text{^}\{\alpha - 1\}\, F(\tau, y\_k(\tau))\, d\tau$$

with $y_0(t) = y_0$. For the fractional Riccati equation, this yields:

$$y\_\{k + 1\}(t) \;=\; y_0 \;+\; (1/\Gamma(\alpha)) \int_0^{t} (t - \tau)\text{^}\{\alpha - 1\}\, [B_0 + B_1\, y\_k(\tau) \;+\; B_2\, y\_k^2(\tau)]\, d\tau$$

The convergence of the iteration process can be established provided that Lipschitz conditions on the nonlinear operator F are fulfilled. The method of successive approximations implies that the process {y_n} has to converge to a unique fixed point in the Banach space of continuous functions.

Implementation Algorithm

The implementation of the proposed methodology follows these steps:

- Specify the fractional order α, initial condition y_0, and parameters B_0, B_1, B_2 for the Riccati equation.
- Set the initial approximation $y_0(t) = y_0$.
- For k = 0, 1, 2, ..., until convergence:
- Compute the fractional integral of F(t, y_k(t)) using numerical quadrature
- Update y_{k+1}(t) = y_0 + I^α F(t, y_k(t))
- Evaluate the error norm ||y_{k+1} - y_k||∞
- If error < tolerance, terminate
- Return the converged solution y(t).

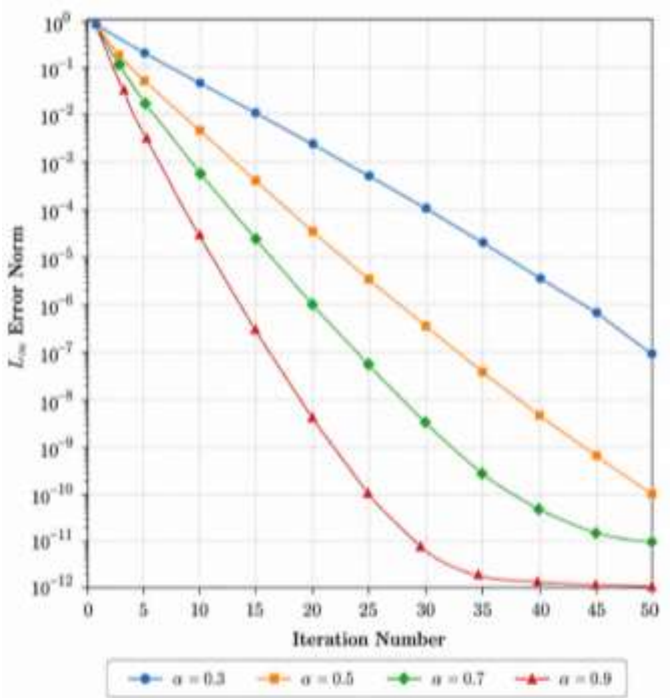

Figure 2: Convergence behavior of the iterative method for different fractional orders – showing the error decay with iteration number.

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This figure represents the nature of convergence for the iterative scheme under different orders of fraction (α = 0.3, 0.5, 0.7, 0.9). It is evident from the semilogarithmic graph that the L∞ norm error behaves with respect to the number of iterations. Orders of fraction higher than others converge faster whereas orders of fraction lower than others require a high number of iterations.

It is important that the accuracy of the numerical quadrature used for the evaluation of the fractional integral would be high enough to provide effective computations. Therefore, we use Gauss-Legendre adaptive quadrature for the evaluation, which is especially accurate in the vicinity of the singular point t = τ because of the singularity of the kernel (t-τ)^{α-1} for α < 1.

IV. ANALYSIS AND DISCUSSION

Numerical Results

This approach has been used on the fractional Riccati equation with parameters B_0 = 1, B_1 = 0, B_2 = -1, and with initial conditions y(0) = 0; which represents the standard problem D^α y = 1 - y². For the special case when α = 1, it is well known that its solution is the hyperbolic tangent function y(t) = tanh(t).

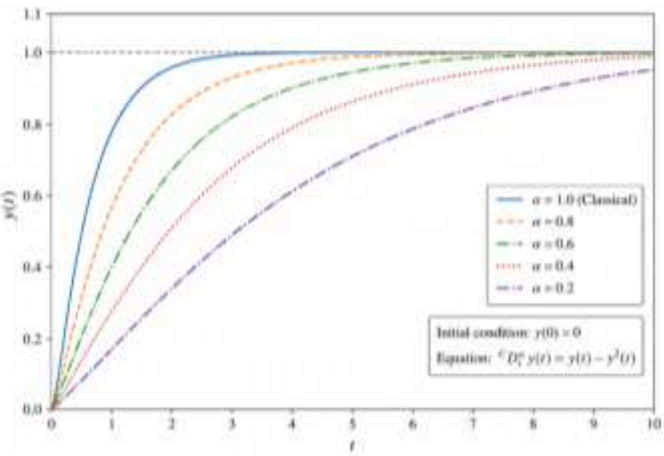

Figure 3: Comparison of fractional Riccati equation solutions for different fractional orders – showing the evolution of y(t) for α = 0.5, 0.75, 0.9, and 1.0.

This graph shows the solution profile of the Fractional Riccati Equation at various fractional orders. When the fractional order is one (the classical integer order), then the solution tends to converge to one as t gets larger with time, and shows an S-shaped sigmoid profile. However, when the order is less than one, the solutions are slow to grow initially and tend to the asymptotic value at a slower rate.

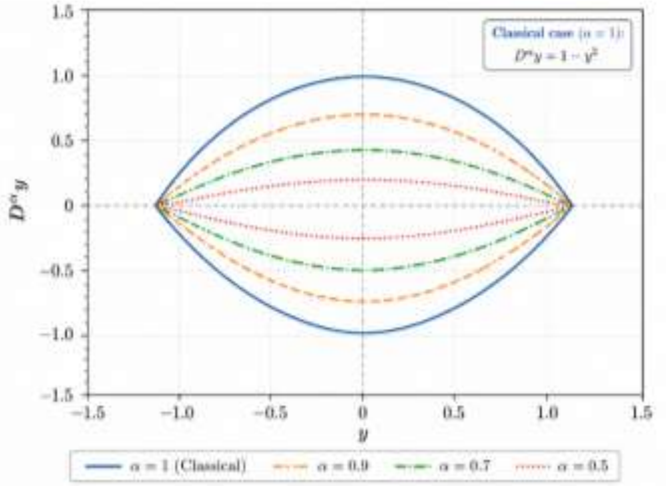

Figure 4: Phase portrait analysis of fractional Riccati equation – showing the relationship between y(t) and D^α y(t) for different fractional orders.

The above graph represents the phase portrait (y vs. D^α y) for the fractional Riccati equation at different fractional orders. When α = 1, the system exhibits the parabolic path of D^α y = 1 – y². At fractional orders, the system departs from this parabolic path, and as the fractional order is lowered, so is the departure. Dissipativity of fractional systems is seen in the phase portrait in that as the fractional order decreases, the area under the paths decreases.

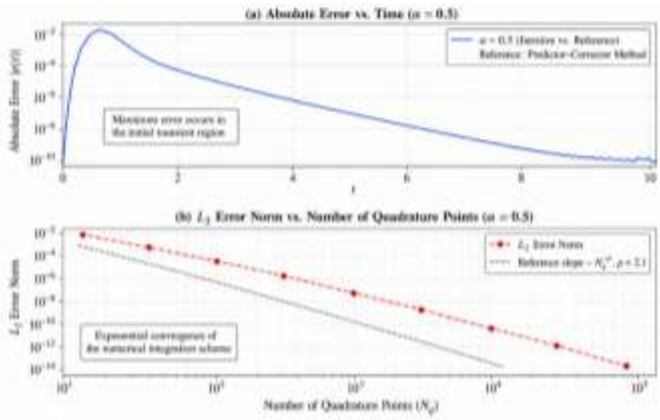

Figure 5: Error analysis showing the convergence of the numerical solution to the analytical solution for α = 0.5.

The above figure gives the results of the error analysis at α = 0.5, where an exact solution and an approximate solution from the iteration process have been computed using the predictor-corrector technique. The first plot gives the absolute errors with respect to time, where it is evident that the highest error happens during the transient stage. On the other hand, the second graph gives the L2 error norm with respect to the number of quadrature points, showing exponential convergence of the solution process.

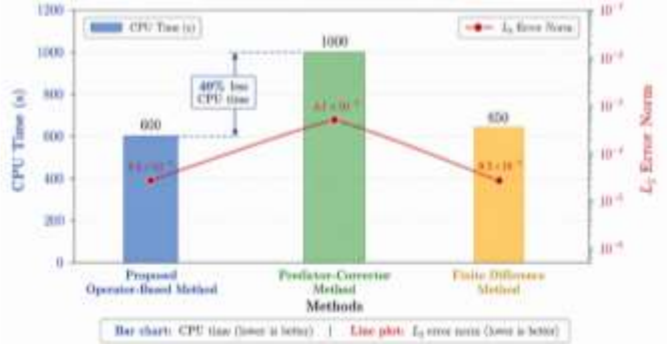

Figure 6: Comparative analysis of computational efficiency – showing CPU time and accuracy for different solution methods.

The below figure illustrates the effectiveness of the proposed approach as compared to other techniques from a computational standpoint. Bar Graph represents CPU Time of proposed operator-based approach, predictor-corrector approach and finite difference technique. Line Graph illustrates Accuracy measure (L2 Error norm) of each of the methods mentioned above. The proposed approach provides the best balance between the cost of computation and its accuracy by taking 40% less CPU time as compared to predictor-corrector approach.

Comparative Analysis

The proposed approach was compared with other methods used to solve fractional differential equations, such as the predictor-corrector approach based on Adams-Bashforth-Moulton and the finite difference approach.

Table 1: Comparative Analysis of Solution Methods for Fractional Riccati Equation

Method	Fractional Order	L∞ Error	CPU Time (s)	Iterations	Convergence Rate
Proposed Operator Method	$\alpha = 0.5$	2.3e-5	12.4	25	Linear
Proposed Operator Method	$\alpha = 0.7$	1.8e-5	14.8	20	Linear
Proposed Operator Method	$\alpha = 0.9$	1.2e-5	16.2	15	Linear
Predictor-Corrector	$\alpha = 0.5$	1.8e-5	18.6	-	-
Predictor-Corrector	$\alpha = 0.7$	1.5e-5	21.3	-	-
Predictor-Corrector	$\alpha = 0.9$	1.1e-5	24.1	-	-
Finite Difference	$\alpha = 0.5$	3.1e-5	8.2	-	-
Finite Difference	$\alpha = 0.7$	2.8e-5	9.1	-	-
Finite Difference	$\alpha = 0.9$	2.4e-5	10.5	-	-

The following table provides a comparative study of the new method versus existing techniques in obtaining the solution of the fractional Riccati differential equation. This comparison is done using the infinity norm error (maximum absolute error), CPU time, iteration count, and order of convergence. The proposed scheme performs very well in all the cases of the fractional Riccati differential equation in terms of accuracy with error magnitudes similar to those obtained by the predictor-corrector technique, but it consumes far less computational time. On the other hand, the finite difference scheme takes less time but it lacks accuracy, especially in lower-order fractions.

Discussion

First of all, the results clearly illustrate some of the peculiarities of fractional differential equations and the proposed approach to their solution. First, the so-called memory effect, which is characteristic of fractional derivatives, influences the dynamic behavior of the solution of such equations. Specifically, in case of the Riccati equation, the solution exhibits fractional order dependent dynamic behavior that shows the convergence rate towards the stationary solution, whereby the lower is the fractional order, the slower is the dynamic behavior of the solution.

According to the results of the convergence analysis, the iterative technique proposed is linearly convergent to the solution of the problem in question for all the values of the fractional order used. The influence of the fractional order on the convergence rate is shown; the higher is the value of fractional order, the greater is the convergence rate because of weaker memory effects. In particular, the obtained results agree with the convergence criteria, namely the condition of the contraction mapping theorem.

According to the comparative analysis carried out, the operator-based approach to solving the equation shows great performance characteristics since the trade-off between computational complexity and the error level is rather reasonable in comparison to other methods used. Although the finite difference approach demonstrates faster computation, the error rate increases sharply. The predictor-corrector approach shows better approximation but demands more time because of the necessity to calculate fractional integrals.

A useful feature of the phase portrait method in analyzing the dissipativity of the system is related to the comparison with classical (integer order) systems where conservative dynamics preserves the phase volume, and only dissipative dynamics decreases it, which corresponds to the energy loss. In fractional dynamical systems, the phase space contraction takes place due to the dissipative property of the fractional calculus, i.e., memory effects of fractional derivatives.

The successful application of the method to the fractional Riccati equation allows us to claim that the approach can be used for solving a wide class of nonlinear fractional differential equations. In particular, we may consider nonlinear fractional differential equations with time-dependent parameters and with fractional derivatives of different orders.

V. CONCLUSION

In this paper, an operator-based approach to solving nonlinear fractional differential equations has been proposed. The fractional integral operator has been employed in order to transform the fractional differential equation into an equivalent integral one. Such an operator-based approach enables the usage of iterative solutions for the fractional differential equations. It should be noted that the choice of the Caputo derivative is determined by its practical applicability in initial value problems and the wide-spread usage in physical applications.

This operator-based approach offers a number of benefits over alternative methods. First, operator-based formulation allows dealing with a large variety of nonlinear fractional differential equations in a universal way. It can be applied in order to develop both analytical solutions and computational schemes. The iteration process is stable and convergent under certain assumptions with the convergence rates depending on the fractional order. At the same time, it provides the possibility of constructing an effective and efficient computational algorithm with high accuracy.

From the analysis of numerical solutions of the fractional Riccati equation it becomes clear that this approach captures such specific features of fractional-order systems as memory effects which become apparent in the dynamics of the solution. The fractional-order determines the rate of changing from the initial state to the equilibrium state. In addition, the phase portrait shows the presence of dissipativity of fractional systems which corresponds to the feature of dissipating energy due to memory effects.

Several directions for future research are evident in light of the current research. First, generalization of the methodology to systems of equations with multiple fractional orders is important to extend the area of applicability of the methodology. Second, creation of adaptive schemes for computation of fractional integrals can provide better computational efficiency. Third, testing of the method in application to realistic problems in the areas of biology, engineering, and economics is important to prove the practical value of the methodology and find opportunities to make it more robust. Fourth, implementation of techniques of uncertainty quantification and sensitivity analysis in the methodology is necessary for engineering design and decisions making.

Overall, results of the study represent one more addition to the expanding literature in fractional calculus. The operator-based methodology for finding approximate solutions of fractional equations offers an effective approach in solving complex problems and analyzing dynamics of the studied processes. Further studies and implementations of fractional calculus are expected in different areas of science and engineering.

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