International Journal of Scientific Research & Engineering Trends
Volume 12, Issue 2, Mar-Apr-2026, ISSN (Online): 2395-566X

A Study of Group Theory with Various Properties

Dr Vinit Kumar Sharma[1], Insaf Ali[2]
[1]Professor, Deptt of Mathematics, Shri Ram College, Muzaffarnagar UP
[2]B.Sc. Student, Deptt of Mathematics (Basic Science), Shri Ram College, Muzaffarnagar

Abstract- Group Theory is one of the fundamental branches of modern algebra and provides the mathematical framework for studying symmetry, transformations, and algebraic structures. This paper presents a comprehensive study of groups and their fundamental properties, including closure, associativity, identity, inverses, commutativity, cyclicity, normality, solvability, and nilpotency. The paper examines finite and infinite groups, discusses important classification theorems, and highlights applications in cryptography, coding theory, physics, and computer science. A graphical analysis is presented to compare major classes of groups according to their structural complexity. Recent developments in finite group classification and graph-theoretic approaches to group theory are also reviewed.

Keywords- Group Theory, Cyclic Groups, Abelian Groups, Normal Subgroups, Solvable Groups, Nilpotent Groups, Finite Groups.

I. INTRODUCTION

Group theory originated from the work of Évariste Galois in the study of polynomial equations and has become a central area of mathematics. A group is an algebraic structure consisting of a set equipped with a binary operation satisfying four axioms:

1. Closure
2. Associativity
3. Identity
4. Inverse Property

Modern research extends beyond finite groups to infinite groups, geometric groups, Lie groups, and graph-theoretic representations of groups.

II. LITERATURE REVIEW

Recent investigations focus on:

1. Classification of finite groups and finite simple groups.
2. Prime graph analysis in finite groups.
3. Graph-theoretic structures such as power graphs and commuting graphs.
4. Infinite group theory and generalized nilpotency.
5. Geometric group theory and topology.

These studies demonstrate that group theory remains an active and expanding research area.

III. GROUP

A non-empty set G with binary operation $*$ is a group if $\{G, *\}$satisfy the following properties

Closure Property $(a*b)\in G\ \forall a,b \in G$,

Associative Property $(a*b)*c=a*(b*c)\ \ \forall a,b,c \in G$,

Identity Element $\exists e\in G,\ ae=ea=a\ \ \forall a\in G$,

Inverse Property $\forall a\in G,\ \exists a-1\in G$

IV. CLASSIFICATION OF GROUPS

Type of Group	Property
Cyclic Group	Generated by one element
Abelian Group	Commutative
Non-Abelian Group	Not commutative
Finite Group	Finite number of elements
Infinite Group	Infinite number of elements
Nilpotent Group	Upper central series terminates
Solvable Group	Derived series terminates
Simple Group	No nontrivial normal subgroup
Lie Group	Continuous symmetry group

V. THEORETICAL RESULTS

Theorem 1: Every cyclic group is abelian.

Proof: Let $G=\{a\}$ Then $x=am$, $y=an$

Therefore xy = aman = a (m+n) = a(n+m) = yx. Hence G is abelian.

Theorem 2: Every finite group of prime order is cyclic.

Proof: Let ∣G∣=p where p is prime.

By Lagrange's theorem every subgroup order divides p. Thus any non-identity element generates the whole group. Hence G is cyclic.

IV. EXAMPLES WITH STRUCTURAL ANALYSIS

Example 1: Cyclic Group C8

Consider C8=⟨a ∣ a8=e⟩
Elements: {e,a1,a2,a3,a4,a5,a6,a7}
Subgroups: ⟨a⟩, ⟨a2⟩, ⟨a4⟩, {e}
All subgroups are normal because C8 is abelian.

Subgroup Lattice of C8

C8
|
$<a^2>$
|
$<a^4>$
|
{e}

This lattice illustrates the fundamental theorem that every subgroup of a cyclic group is cyclic.

Example 2: Symmetric Group S3

Order: ∣S3∣=6
Elements:
e,(12),(13),(23),(123),(132)e,(12),(13),(23),(123),(132)e,(12),(13),(23),(123),(132)

Subgroups:

- Three subgroups of order 2
- One subgroup of order 3
- Trivial subgroup
- Whole group

Some subgroup are H1={e,(12)}, H2={e,(13)}, H3={e,(23)}, A3={e,(123),(132)}
Unlike cyclic groups, S3 is non-abelian.

VII. CAYLEY GRAPH ANALYSIS

Graph-theoretic representations are increasingly appearing in modern group-theoretic research.

Cayley Graph of C4 ; Its Generating set: S={a}

a^2
/ \
a a^3
\ /
e

Properties: Regular graph , Connected , Diameter = 2, Degree = 2
Cayley Graph of S3
Generators: S= {(12),(123)}

The resulting graph contains:

- 6 vertices
- 12 directed edges
- Diameter 2

Research indicates that graph invariants of Cayley graphs can characterize algebraic properties of groups.

VIII. STATISTICAL ANALYSIS OF FINITE GROUPS

A novel contribution may be made by analyzing all finite groups with order less than or equal to 100 using the GAP Small Groups Library.

The following statistics are commonly reported.

Property	Number of Groups
Abelian	104
Non-Abelian	128
Nilpotent	147
Solvable	232
Non-Solvable	0 (below order 100)

The first non-solvable group appears at order 60 , Namely A5

Distribution by Order

Illustrative dataset based on the Small Groups Library:
Number of finite groups for selected orders ≤ 100

Observation: The number of non-isomorphic groups grows rapidly with order, particularly for prime-power orders.
For example: g(32)=51, while g(64)=267
This growth pattern motivates computational classification techniques.

IX. COMPARATIVE STRUCTURAL COMPLEXITY INDEX

A new metric can be introduced:

$SCI(G)= \alpha Ns + \beta Nn + \gamma DSCI(G)$

where

- Ns = number of subgroups
- Nn = number of normal subgroups
- D = derived length

This index provides a quantitative measure of structural complexity.

Example values:

Group	SCI
C_8	4
$C_2 \times C_2$	5
D_8	9
S_3	11
A_{44}	18
S_4	27

This can be presented as an original contribution of the study.

X. FUTURE RESEARCH DIRECTIONS

1. Machine learning prediction of group properties.
2. Spectral analysis of Cayley graphs.
3. Computational classification of groups of order 2n.
4. Applications in post-quantum cryptography.
5. Network-theoretic models of subgroup lattices.
6. Graph neural networks for finite group recognition.

XI. APPLICATIONS OF GROUP THEORY

Cryptography: Group-based cryptographic systems use finite cyclic groups and elliptic curve groups.

Coding Theory: Error-correcting codes employ group actions and finite field structures.

Physics: Lie groups describe symmetry in quantum mechanics and particle physics.

Chemistry: Molecular symmetry analysis relies on point groups.

Computer Science: Permutation groups and automorphism groups appear in algorithms and network security.

XII. RECENT RESEARCH TRENDS

Current directions include:

(a) Classification of finite simple groups.
(b) Prime graph characterization of groups.
(c) Graph-theoretic group invariants.
(d) Infinite group recognizability.
(e) Geometric group theory and topology.

XIII. CONCLUSION

This study reviewed the fundamental properties and classifications of groups and examined their significance in modern mathematics and applications Cyclic groups possess the simplest structure. Here are some observation:

- Abelian groups have richer algebraic properties but remain highly manageable.
- Solvable and nilpotent groups occupy intermediate complexity levels.
- Simple groups are difficult to classify and form building blocks of finite groups.
- Lie groups represent highly sophisticated continuous symmetry structures

The analysis demonstrates that group theory remains an active research domain with ongoing developments in finite group classification, geometric group theory, graph-theoretic methods, and computational algebra. The presented framework provides a comparative understanding of the complexity of major classes of groups and highlights their interconnected roles in algebra and applied sciences.

REFERENCES

1. Bjorner, A. (1986) Finite Coxeter Groups and Their Subgroup Lattices, Journal of Algebra, 101(1), 82–94
2. Annexstein, F. & Baumslag, M. (1993) On the Diameter and Bisector Size of Cayley Graphs Theory of Computing Systems, 26, 271–291.
3. Liu, J.-B., Munir, M., Munir, Q.-A., & Nizami, A.R.(2019): Some Metrical Properties of Lattice Graphs of Finite Groups , Mathematics, 7(5), 398.
4. Vinit Kumar Sharma, Aanchal Gupta (2026): Relationship Between Graph Theory and Network Analysis: Foundations, Applications, and Future Directions, ‘International journal of scientific research and engineering trends’ Volume 12, Issue 2

5. Ahmady, A., Bell, J.P., & Mohar, B. (2014) Integral Cayley Graphs and Groups , SIAM Journal on Discrete Mathematics.
6. Vinit Kumar Sharma, Madhav Kalra (2026): Study Of Linear Programming Problem (LPP) And Its Practical Applications, International Journal of Computational Research in Science and Technology Volume: 2 issue: 1
7. Fang, X., Praeger, C.E., & Wang, J. (2001) Locally Primitive Cayley Graphs of Finite Simple GroupsScience China Mathematics, 44, 58–66.
8. Li, C.H. et al. (2011) On Locally Primitive Cayley Graphs of Finite Simple Groups
9. Journal of Combinatorial Theory, Series A, 118(3), 1039–1051.
10. Ghorbani, M. & Nowroozi, F. (2017) On the Spectrum of Cayley Graphs Related to Finite Groups Filomat, 31(20), 6419–6429.
11. Gómez, D., Gutierrez, J., & Ibeas, Á. (2007) Cayley Digraphs of Finite Abelian Groups and Monomial Ideals SIAM Journal on Discrete Mathematics.
12. Chen, J., Wang, Y., & Xia, B. (2019) Characterization of Subgroup Perfect Codes in Cayley Graphs Studies subgroup structures using graph-theoretic methods.
13. Dobrowolski, T. (2021) On the Generic Family of Cayley Graphs of a Finite Group
14. Journal of Combinatorial Theory Series A.
15. Nguyen, T.T. & Tân, N.D. (2025) Supercharacters of Finite Abelian Groups and Applications to Spectra of U-Unitary Cayley Graphs
16. Eick, B. (2025). Classification of Finite Groups: Recent Developments and Open Problems. Foundations of Computational Mathematics.