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Weak Insertion of a Baire-One Function

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Abstract **–** A sufficient condition in terms of lower cut sets are given for the weak insertion of a Baire-one function between two comparable real-valued functions on the topological spaces that $\Lambda-$sets are $G\delta-$sets.

Keywords **–** Weak Insertion, Strong Binary Relation, Baire-One Function, $\Lambda-$Sets, Lower Cut Set.

I. Introduction

A generalized class of closed sets was considered by Maki in 1986 [5]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [5].

Results of Katětov [2] , [3] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a necessary and sufficient condition for the insertion of a Baire-one function between two comparable real-valued functions on the topological spaces that $\Lambda-$sets are $G_\delta-$sets.

A real-valued function f defined on a topological space *X* is called Baireone if the preimage of every open subset of $\mathbb{R}$ is a $F_\sigma-$set in *X*.

If g and f are real-valued functions defined on a space X, we write $g \le f$ in case $g(x) \le f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [4].

A property P defined relative to a real-valued function on a topological space is a B_1-property provided that any constant function has property P and provided that the sum of a function with property P and any Baire-one function also has property P. If P_1 and P_2 are B_1-properties, the following terminology is used: A space X has the weak B_1-insertion property for (P_1, P_2) iff for any functions g and *f* on *X* such that $g \le f$, *g* has property P_1 and *f* has property P_2, then there exists a Baire-one function h such that $g \le h \le f$.

In this paper, for a topological space that $\Lambda-$sets are $G_\delta-$sets, is given a sufficient condition for the weak B_1-insertion property. Also several insertion theorems are obtained as corollaries of these results.

II. The Main Result

Before giving a sufficient condition for insertability of a Baire-one function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$. A^Λ is called kernel of A.

The following first two definitions are modifications of conditions considered in [2], [3].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows:

$x \,\bar{\rho}\, y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set P (X) of a topological space X is called a strong binary relation in P (X) in case ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in P (X) such that $A_i \rho C$ and $C \rho B_j$ for any $i \in\{1,...,m\}$and any $j \in\{1,..., n\}$.

2) If $A \subseteq B$, then $A \,\bar{\rho}\, B$.

3) If $A\rho B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < l\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \le l\}$ for a real number l, then $A(f, l)$ is called a lower indefinite cut set in the domain of f at the level l.

Theorem 2.1. Let g and f be real-valued functions on the topological space X, that $\Lambda-$sets in X are $G_\delta-$sets, with $g \le f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a Baire-one function h defined on X such that $g \le h \le f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \le f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by F (t)= A(f, t) and G(t)= A(g, t). If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \,\bar{\rho}\, F(t_2)$, $G(t_1) \,\bar{\rho}\, G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and

2 of [3] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then F (t_1) ρH(t_2), H(t_1) ρH(t_2) and H(t_1) ρG(t_2).

For any x in X, let h(x) = inf{t ∈ ℚ : x ∈ H(t)}.

We first verify that $g \le h \le f$: If *x* is in *H*(*t*) then *x* is in G(t') for any t' > t; since x in G(t')= A(g, t') implies that g(x) ≤ t', it follows that g(x) ≤ t. Hence g ≤ h. If x is not in H(t), then x is not in F (t') for any t < t; since x is not in F (t')= A(f, t') implies that f(x) >t', it follows that f(x) ≥ t. Hence h ≤ f.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is a F_σ−set in X, i.e., h is a Baire-one function on X. •

The above proof used the technique of theorem 1 of [2].

III. Applications

Definition 3.1. A real-valued function f defined on a space X is called upper semi-Baire-one (resp. lower semi-Baire-one) if $f^{-1}(-\infty,t)$ (resp. $f^{-1}(t,+\infty)$) is a F_σ−set for any real number t.

The abbreviations usc, lsc, usB_1 and lsB_1 are used for upper semicontinuous, lower semicontinuous, upper semi-Baire-one and lower semi-Baire-one,respectively.

Remark 1. [2], [3]. A space X has the weak c−insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of theorem 2.1, we suppose that X is a topological space that Λ−sets are G_δ−sets.

Corollary 3.1. For each pair of disjoint G_δ−sets G_1, G_2, there are two F_σ−sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ if and only if X has the weak B_1−insertion property for (usB_1, lsB_1).

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB1, g is usB_1, and g ≤ f. If a binary relation ρ is defined by AρB in case $A^\Lambda \subseteq B^V$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

A(f, t_1) ⊆{x ∈ X : f(x) ≤ t_1}⊆{x ∈ X : g(x) < t_2}⊆ A(g, t_2);

since {x ∈ X : f(x) ≤ t_1} is a G_δ−set and since {x ∈ X : g(x) < t_2} is a F_σ−set, it follows that $A(f, t_1)^\Lambda \subseteq A(g, t_2)^v$. Hence $t_1 < t_2$ implies that A(f, t_1) ρA(g, t_2). The proof follows from Theorem 2. 1.

On the other hand, let G_1 and G_2 are disjoint G_δ−sets. Set $f = \chi G_1^c$ and g = χG_2 , then f is lsB1,g is usB_1, and g ≤ f. Thus there exists Baire-one function h such that g ≤ h ≤ f. Set F_1 = {x ∈ X : h(x) <1/2} and F_2 = {x ∈ X : h(x) > 1/2}, then F_1 and F_2 are disjoint F_σ−sets such that G_1 ⊆ F1 and $G_2 \subseteq F_2$. •

Remark 2. [6]. A space X has the weak c−insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. For every F of F_σ−set, F^Λ is a F_σ−set if and only if X has the weak B1−insertion property for (lsB_1, usB_1).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For every F of F_σ−set we have F^Λ is a F_σ−set.

(ii) For each pair of disjoint F_σ−sets as F_1 and F_2 we have $F_1^\Lambda \cap F_2^\Lambda = \emptyset$.

The proof of lemma 3.1 is a direct consequence of the definition Λ−sets.

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and f ≤ g. If a binary relation ρ is defined by AρB in case $A^\Lambda \subseteq F \subseteq F^\Lambda \subseteq B^V$ for some F_σ−set F in X, then by hypothesis and lemma 3.1 ρ is a strong binary relation in the power set of X. If t1 and t_2 are any elements of ℚ with $t_1 < t_2$, then

A(g, t_1)= {x ∈ X : g(x) < t_1}⊆{x ∈ X : f(x) ≤ t_2};

= A(f, t_2);

since {x ∈ X : g(x) < t_1} is a F_σ−set and since {x ∈ X : f(x) ≤ t_2} is a G_δ−set, by hypothesis it follows that A(g, t_1) ρA(f, t_2). The proof follows from Theorem 2.1.

On the other hand, Let F_1 and F_2 are disjoint F_σ−sets. Set f = χF_2 and $g = \chi F_1^c$ then f is lsB_1, g is usB_1, and f ≤ g.

Thus there exists Baire-one function h such that f ≤ h ≤ g. Set G_1 = {x ∈ X : h(x) ≤ 1/3} and G_2 = {x ∈ X : h(x) ≥ 2/3} then G1 and G2 are disjoint Gδ−sets such that F1 ⊆ G_1 and $F_2 \subseteq G_2$. Hence $F_1^\Lambda \cap F_2^\Lambda = \emptyset$. •

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